Tight Revenue Gaps among Multi-Unit Mechanisms

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This paper considers Bayesian revenue maximization in the k-unit setting, where a monopolist seller has k copies of an indivisible item and faces n unit-demand buyers (whose value distributions can be non-identical). Four basic mechanisms among others have been widely employed in practice and widely studied in the literature: Myerson Auction, Sequential Posted-Pricing, $(k+1)$-th Price Auction with Anonymous Reserve, and Anonymous Pricing. Regarding a pair of mechanisms, we investigate the largest possible ratio between the two revenues (a.k.a. the revenue gap), over all possible value distributions of the buyers.

Divide these four mechanisms into two groups: (i) the discriminating mechanism group, Myerson Auction and Sequential Posted-Pricing, and (ii) the anonymous mechanism group, Anonymous Reserve and Anonymous Pricing. Within one group, the involved two mechanisms have an asymptotically tight revenue gap of $1 + \Theta(1/\sqrt{k})$. In contrast, any two mechanisms from the different groups have an asymptotically tight revenue gap of $\Theta(\log k)$.

CCS Concepts: • **Theory of computation** → **Algorithmic mechanism design**; **Computational pricing and auctions**.

Additional Key Words and Phrases: Bayesian Revenue Maximization, Myerson Auction, Sequential Posted Pricing, Anonymous Reserve, Anonymous Pricing, Revenue Gap

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1 INTRODUCTION

"Simple vs. optimal" is one of the central themes in Bayesian mechanism design. The revenue-optimal mechanisms are more of theoretical significance, but are usually complicated and hard to implement in practice. On the other hand, most of the commonly used mechanisms in real life are much simpler, although sacrificing a (small) amount of revenue. This trade-off motivates the study on how well simple mechanisms can approximate the optimal mechanisms.

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Even in the most basic single-item model, the optimal mechanism is already complicated. This mechanism is proposed in the seminal work by Myerson [39]. It needs full knowledge of all buyers' individual value distributions. Typically, the value distributions are estimated from market research and historical transaction records. Hence, the knowledge can only be "probably approximately correct" (especially in large markets) and the optimal mechanism is very fragile to the estimation errors. Also, Myerson's auction involves price discrimination among buyers, which is not allowed in most real businesses.

Simple mechanisms can address the above issues and are prevalent in both online and offline shops. For example, consider selling a number of identical copies of some product on Amazon. (This is captured by the *multi-unit* model, see Section 1.1 for details.) The seller simply posts a price, and a buyer decides to buy one copy if the price is acceptable to him. This mechanism is called *anonymous pricing*. In order to find the optimal price, the seller only needs to know the overall demand, which is much easier to estimate than the individual value distributions as in Myerson's auction. Once again, the question is how well the anonymous pricing mechanism can approximate the optimal revenue.

The last two decades have seen extensive progress on the "simple versus optimal" trade-off [2, 3, 6, 8, 13–15, 19, 21, 23, 25, 28–30, 32]. By now we can say that it constitutes a subfield within mechanism design. In this work, we will study this trade-off in the multi-unit model.

1.1 Background

Let us first review the previous results. In the most basic *single-item* model, four fundamental mechanisms among others are widely studied. Denote by $\mathbf{F} = \{F_j\}_{j \in [n]}$ the independent value distributions of buyers $j \in [n]$. These four mechanisms work as follows (see Section 2.2 for the formal definitions).

- Anonymous Pricing (AP): This mechanism treats all buyers equally by posting a price p. Upon arrival, a buyer will pay this price p and take the item, when his value $b_j \sim F_j$ is higher than p (and the item is still available). If the seller knows the value distributions $\{F_j\}_{j \in [n]}$, she would select a particular price p to maximize her expected revenue among all Anonymous Pricing mechanisms.
- Sequential Posted Pricing (SPM): This mechanism selects an array of prices $\{p_j\}_{j \in [n]}$ and an ordering $\sigma : [n] \mapsto [n]$. The buyers join in the mechanism *sequentially* $\sigma(1), \cdots, \sigma(n)$, and each index-$\sigma(j)$ buyer must pay the *order-specific* price p_j if winning. This discrimination can give better revenue than Anonymous Pricing.
- Anonymous Reserve (AR): This is a variant of the Second-Price Auction. The seller ignores the buyers whose bids b_j are below an anonymous reserve r. The winner (which exists only if the highest bid $b_{(1)}$ is above the reserve r) is the highest of the remaining buyers, and his payment is the bigger one between the second highest bid $b_{(2)}$ and the reserve r.
- Myerson Auction (OPT): A generic auction $\mathcal{A} : \{b_j\}_{j \in [n]} \mapsto (\mathbf{x}, \boldsymbol{\pi})$ is a mapping from the bids/values to the allocations $\mathbf{x} = (x_j)_{j \in [n]}$ and the payments $\boldsymbol{\pi} = (\pi_j)_{j \in [n]}$. In the single-item case, Myerson Auction is the optimal one among those mappings [39]. (When the distributions $\{F_j\}_{j \in [n]}$ are identical, Myerson Auction degenerates to Anonymous Reserve.)

These four mechanisms together form the hierarchy in Figure 1, where each arrow goes from a more complicated mechanism with higher revenue to a simpler mechanism with lower revenue. There are two notable distinctions among the four mechanisms.

- Anonymity (AP and AR) vs. Discrimination (SPM and OPT). We say a mechanism is *discriminating* if, when different buyers become the winner, the required payments can be

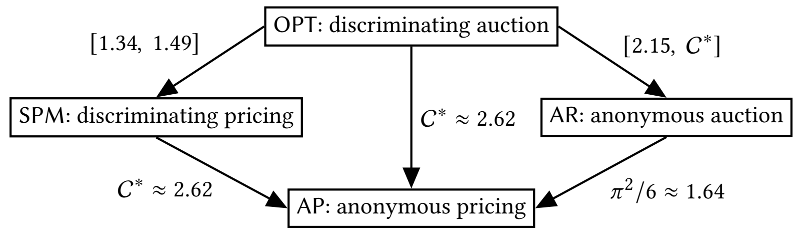


Fig. 1. Demonstration for the **previous results** in the **single-item** setting with asymmetric regular buyers, where an interval indicates the best known lower/upper bounds, and a number indicates a tight bound. For the references of these results and further discussions, one can refer to [31, Section 6] and [27, Chapter 4].

different. Otherwise we say the mechanism is *anonymous*. Intuitively, discrimination gives a mechanism more power to extract revenue.

- Pricing (AP and SPM) vs. Auction (AR and OPT). In a pricing scheme, the buyers simply make take-it-or-leave-it decisions based on the given prices. In contrast, an auction is an arbitrary mapping from the bids to the allocations and the payments. Auctions can gain higher revenues than pricing schemes by further leveraging the competition among buyers.

Because SPM is a discriminating pricing scheme and AR is an anonymous auction, they have different powers and are incomparable. Accordingly, there are five comparable mechanism pairs (i.e., the five arrows in Figure 1).

To understand the relative powers of those mechanisms, the very first question is how large the *revenue gap* between any two mechanisms can be. We characterize the revenue gap as the *approximation ratio*[1] between the two revenues. Formally, for a more complicated mechanism $\mathcal{M}_1$ and a simpler mechanism $\mathcal{M}_2$, their approximation ratio is given by

$$\mathfrak{R}_{\mathcal{M}_1/\mathcal{M}_2} \overset{\text{def}}{=} \sup\left\{ \frac{\text{Rev}_{\mathcal{M}_1}(\mathbf{F})}{\text{Rev}_{\mathcal{M}_2}(\mathbf{F})} \,\middle|\, \mathbf{F} \in \mathcal{F} \right\},$$

where $\text{Rev}_{\mathcal{M}}(\mathbf{F})$ denotes the revenue from a mechanism $\mathcal{M}$ on an input instance $\mathbf{F} = \{F_j\}_{j \in [n]}$, and the supremum is taken over a certain family of distributions $\mathbf{F} \in \mathcal{F}$.

For the single-item model, the known results are shown in Figure 1. Notice that all these revenue gaps are universal constants, and most of them have matching lower and upper bounds.

From Single Unit to Multiple Units. In this work, we focus on the *k-unit setting*, where the seller has $k \geq 1$ identical copies of an item, and aims to sell them to n unit-demand buyers. This setting is much more realistic and common in real business. Further, it is of intermediate complexity in comparison with the (more restricted) single-item setting and the (more general) multi-item setting.[2] Nonetheless, the "simple vs. optimal" trade-offs are much less understood in this setting than in the single-item setting.

Since the k-unit setting is still a *single-parameter* setting, Myerson Auction remains revenue-optimal [39]. In addition, both of Anonymous Pricing and Sequential Posted Pricing can be naturally extended to this setting. For Anonymous Reserve, the counterpart auction is no longer "second-price-type", but is the $(k + 1)$-th Price Auction with Anonymous Reserve.

[1]The earlier "mechanism design for digit goods" literature [6, 13, 14, 23, 25], due to technical reasons, often uses the term "competitive ratio" rather than "approximation ratio".

[2]In the k-unit setting, the k copies are *identical*. But in the multi-item setting, the items can be *heterogeneous*.

1.2 An overview of our results

In the k-unit setting, previously only the revenue gap $\mathfrak{R}_{\mathsf{OPT/SPM}}$ between OPT and SPM is well understood [2, 7, 41], but the other four gaps are widely open. By exploring the relative power of those mechanisms systematically, in this work we establish the (asymptotically) tight ratios of all previously unknown revenue gaps. We formalize our new results as the next two theorems and demonstrate them in Figure 2. (Therein, the regularity assumption is very standard in the mechanism design literature [39]; see Section 2.1 for its definition.)

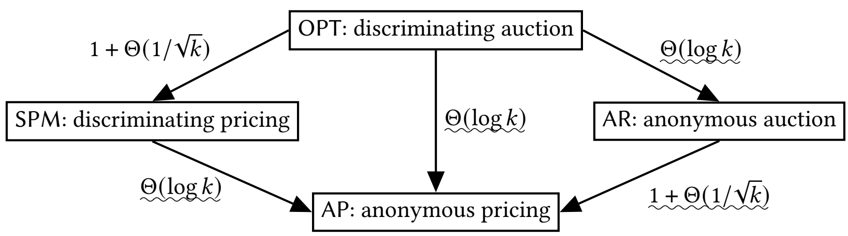


Fig. 2. Demonstration for the revenue gaps among basic mechanisms in the k-**unit** setting, given that the value distributions are regular. Our new results are underwaved. The $1+\Theta(1/\sqrt{k})$ approximation result between AR and AP is given in Theorem 1, and the other three results are given in Theorem 2.

Theorem 1 (Anonymous Reserve vs. Anonymous Pricing). *For the unit-demand buyers $j \in [n]$, in each of the following three settings,*[3] *the revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ between* Anonymous Reserve *and* Anonymous Pricing *is $\mathfrak{R}_{\mathsf{AR/AP}}(k) = 1+\Theta(1/\sqrt{k})$:*

(1) *The asymmetric general setting, where the buyers have independent but not necessarily identical value distributions.*
(2) *The i.i.d. general setting, where the value distributions are identical.*
(3) *The asymmetric regular setting, where the value distributions are regular but not necessarily identical.*

Theorem 2 (Discriminating Mechanisms vs. Anonymous Mechanisms). *When the unit-demand buyers $j \in [n]$ have independent and regular value distributions, each of the next three revenue gaps is of order $\Theta(\log k)$:*

(1) *The revenue gap $\mathfrak{R}_{\mathsf{OPT/AP}}(k)$ between* Myerson Auction *and* Anonymous Pricing.
(2) *The revenue gap $\mathfrak{R}_{\mathsf{SPM/AP}}(k)$ between* Sequential Posted Pricing *and* Anonymous Pricing.
(3) *The revenue gap $\mathfrak{R}_{\mathsf{OPT/AR}}(k)$ between* Myerson Auction *and* Anonymous Reserve.

Similar to the AR vs. AP revenue gap, the previous works [2, 7, 41] show that the OPT vs. SPM revenue gap is also of order $1+\Theta(1/\sqrt{k})$. Consequently, regarding the discriminating mechanism group (OPT and SPM) and the anonymous mechanism group (AR and AP), each revenue gap across these two groups is $\Theta(\log k)$, but the revenue gap between the two mechanisms in one group tends to vanish (at the rate of $1/\sqrt{k}$) when the number of copies $k \in \mathbb{N}_{\geq 1}$ becomes large. These messages can be easily inferred from Figure 2.

As mentioned, the revenue gaps identify the power and the limit of "discrimination vs. anonymity" and "auction vs. pricing" in revenue maximization. Different from the single-item setting, where all the revenue gaps are universal constants (see Figure 1), our new results in the k-unit setting are more informative. When the number of copies $k \in \mathbb{N}_{\geq 1}$ is large:

[3] In the i.i.d. regular setting, an asymptotically tight bound $1+\Theta(1/\sqrt{k})$ is shown in [41, Section 4.2], [27, Section 4.5] and [7, Section 5].

- Auctions are not much more helpful than pricing schemes in extracting the revenue (i.e., just an $1 + \Theta(1/\sqrt{k})$ improvement), no matter whether discrimination is allowed or not.
- Discrimination is always very useful, and can even give an unbounded improvement (up to a $\Theta(\log k)$ factor) on the revenue.

These propositions meet what we observe in real business: auctions are rarely used in practice, whereas different kinds of price discrimination are rather common.

1.2.1 First Result: Anonymous Reserve vs. Anonymous Pricing. In this section, we sketch the proof of our $1 + \Theta(1/\sqrt{k})$ approximation result for the AR vs. AP revenue gap (Theorem 1). In fact, we can represent the exact ratio $\mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}$ as an explicit integration formula, (although this formula in general does not admit an elementary expression). We acquire this formula by solving a mathematical programming generalized from [32, Program (4)], which resolves the same problem for the single-item case $k = 1$.

However, many crucial properties of the single-item case do not preserve in the general case $k \geq 1$. In the single-item case, Anonymous Reserve relies on the first/second order statistics $b_{(1)}$ and $b_{(2)}$ (i.e., the biggest and second biggest sampled bids/values), and Anonymous Pricing relies on the $b_{(1)}$. Therefore, we only need to reason about these two random variables, $b_{(1)}$ and $b_{(2)}$, together with the correlation between them. In the k-unit case, however, up to $(k+1)$ random variables $b_{(1)}, \cdots, b_{(k+1)}$ must be taken into account, and the correlation among them becomes much more complicated.

For the above reasons, we cannot modify and re-adopt the approach of the work [32] in a naive way. Instead, with the purpose of handling the highly correlated order statistics $b_{(i)}$'s, we will develop a new structural lemma about the *Poisson binomial distributions* (PBDs). This new lemma mainly relies on the *log-concavity* of the PBDs.

Lemma (Bernoulli Sum Lemma). *Given two arrays of Bernoulli random variables: $\{X_j\}_{j \in [n]}$ are i.i.d., while $\{Y_j\}_{j \in [n]}$ are independent yet not necessarily identically distributed. For the random sums $X = \sum_{j \in [n]} X_j$ and $Y = \sum_{j \in [n]} Y_j$, there exists some threshold $s \in \mathbb{R}$ such that:*

(1) $\Pr[X \leq t] \geq \Pr[Y \leq t]$ *for any $t < s$.*
(2) $\Pr[X \leq t] \leq \Pr[Y \leq t]$ *for any $t \geq s$.*

With the help of this lemma, we can characterize the worst-case instance of the mentioned mathematical programming, for $k \geq 1$ and $n \geq 1$. To this end, let us formulate the AR and AP revenues. Denote by F_j the cumulative distribution function (CDF) of buyer j's value, and D_i the CDF of the i-th order statistic $b_{(i)}$. The Anonymous Reserve revenue (Fact 3) is given by

$$\mathsf{AR}(r) \ = \ \mathsf{AP}(r) \ + \ k \cdot \int_r^\infty (1 - D_{k+1}(x)) \cdot \mathrm{d}x, \qquad \forall r \geq 0.$$

where $\mathsf{AP}(r)$ is the revenue by posting the price $p = r$ in Anonymous Pricing. Further, the AP revenue (Fact 2) depends on the top-k CDF's $\{D_i(r)\}_{i \in [k]}$ at this reserve $r \geq 0$.

Now consider a Bernoulli sum $Y = \sum_{j \in [n]} Y_j$, for which the individual failure probabilities are $\Pr[Y_j = 0] = F_j(r)$. This choice of the failure probabilities ensures $\Pr[Y \leq i - 1] = D_i(r)$ for every $i \geq 1$. Further, we can find another array of i.i.d. Bernoulli random variables $\{X_j\}_{j \in [n]}$ so that the sum $X = \sum_{j \in [n]} X_j$ satisfies

$$\Pr[X \leq k] \ = \ \Pr[Y \leq k] \ = \ D_{k+1}(r).$$

(The existence of such $\{X_j\}_{j \in [n]}$ is obvious.) Then our Bernoulli Sum Lemma shows that

$$\Pr[X \leq i - 1] \ \geq \ \Pr[Y \leq i - 1] \ = \ D_i(r)$$

for each $i \in [k]$, where the equality holds when the $\{Y_j\}_{j \in [n]}$ are also i.i.d.

Informally speaking, the above inequalities and the equality condition imply that, the ratio $\mathsf{AR}(r)/\mathsf{AP}(r)$ is maximized when the value CDF's are equal $F_1(r) = \cdots = F_n(r)$ at this reserve. Following this argument and with extra efforts, we have the next observation.

Observation. *For each $k \geq 1$ and $n \geq 1$, the worst case for the $\mathfrak{R}_{\mathsf{AR/AP}}$ revenue gap happens when the value distributions are identical, i.e., $\mathbf{F}^* = \{F^*\}^n$, (although this worst-case common distribution F^* is given by an implicit equation and does not admit an elementary expression).*

Furthermore, it is noteworthy that the above approach enables a unified constructive proof for the upper-bound/lower-bound parts of the general case $k \geq 1$. In contrast, the former work [32] establishes these two parts of the single-item case separately, and their upper-bound proof is non-constructive.

Our Bernoulli Sum Lemma can find its applications in related directions. As mentioned, we leverage it mainly to handle the order statistics. Apart from the "simple vs. optimal mechanism design" paradigm, on other topics such as "learning simple mechanisms from samples" [9, 14, 33, 37, 38], the order statistics are also of fundamental interests. Conceivably, our new lemma would be helpful for those topics, in a similar manner as this paper.

1.2.2 Second Result: Discriminating Mechanisms vs. Anonymous Mechanisms. In this section we sketch the proof of Theorem 2, which claims that the revenue gaps $\mathfrak{R}_{\mathsf{OPT/AP}}$, $\mathfrak{R}_{\mathsf{SPM/AP}}$ and $\mathfrak{R}_{\mathsf{OPT/AR}}$ are all of order $\Theta(\log k)$. In fact, any one bound implies the other two. This is because the revenue gaps within the discriminating/anonymous groups (OPT vs. SPM, and AR vs. AP) are both *constants* $1 + \Theta(1/\sqrt{k}) = \Theta(1)$, and these constants are dominated by the $\Theta(\log k)$ bound.

For these reasons, it suffices to only prove the OPT vs. AP revenue gap $\mathfrak{R}_{\mathsf{OPT/AP}} = \Theta(\log k)$. Actually, an $\Omega(\log k)$ lower bound for this revenue gap is already shown in [28, Example 5.4], so we only need to prove the $O(\log k)$ upper bound.

We actually prove the $O(\log k)$ upper bound between Anonymous Pricing and a benchmark called Ex-Ante Relaxation (EAR in short). It is known that this benchmark always exceeds the Myerson Auction revenue [11]. To acquire the $O(\log k)$ upper bound, we will start with a mathematical programming generalized from [3, Equations (1) and (2)].

However, the general-case mathematical programming has a very different structure as it is in the single-item case. When $k = 1$, the worst-case instance (i.e., the optimal solution, see [3, Section 4.3]) turns out to be a *continuum of "small" buyers* – any single buyer has an infinitesimal contribution to the EAR benchmark, but there are infinitely many buyers $n \to \infty$ (in the sense of large markets [4, 36]). Accordingly, it is better to think about the "density" of different types of buyers, instead of the number of buyers.

But in the general case, the $\Omega(\log k)$ lower-bound instance [28, Example 5.4] essentially is constituted by "big" buyers – a certain amount of buyers contribute at least $1/k$ unit to the EAR benchmark each, while every other buyer contributes strictly 0 unit and can be omitted. More importantly (see Remark 4), if we insist on a continuum of "small" buyers in the general case $k \geq 1$, then the EAR vs. AP revenue gap turns out to be (at most) a universal constant for whatever $k \geq 1$.

For these reasons, the current approach must be very different from the single-item case. At a high level, to handle the general case $k \geq 1$, we will classify the buyers $j \in [n]$ into groups, and then bound the individual contributions from these groups to the EAR benchmark.

In more details, we can employ the technique developed in [3, Lemma 4.1], and thus transform the mentioned mathematical programming into the following one.

Variables:

- $\{v_j\}_{j\in[n]} \in \mathbb{R}^n_{\geq 0}$, where $v_j = \arg\max\{p \cdot (1-F_j(p)) : p \geq 0\}$ for $j \in [n]$, are the *buyer-wise optimal posted prices* of the distributions $\mathbf{F} = \{F_j\}_{j\in[n]}$.
- $\{q_j\}_{j\in[n]} \in [0,1]^n$, where $q_j = 1 - F_j(v_j)$ for $j \in [n]$, are the *buyer-wise optimal quantiles.*
- The resulting $\{v_j q_j\}_{j\in[n]} \in \mathbb{R}^n_{\geq 0}$ are the *buyer-wise optimal revenues.*

Constraints:

- The capacity constraint, $\sum_{j\in[n]} q_j \leq k$.
- The feasibility constraint, $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in \mathbb{R}_{\geq 0}$.

Objective: Maximize the Ex-Ante Relaxation benchmark $\mathsf{EAR}(\mathbf{F}) = \sum_{j\in[n]} v_j q_j$.

Regarding the EAR benchmark, the buyer-wise optimal revenues $\{v_j q_j\}_{j\in[n]}$ are precisely the individual contributions from the distributions $\{F_j\}_{j\in[n]}$. Given the capacity constraint (in the sense of the Knapsack Problem), the quantiles $\{q_j\}_{j\in[n]}$ can be viewed as the individual capacities. Therefore, the prices $\{v_j\}_{j\in[n]}$ can be viewed as the *bang-per-buck ratios* (i.e., the contribution to the EAR benchmark per unit of the capacity).

To find the optimal solution, of course we prefer those distributions with higher bang-per-buck ratios $\{v_j\}_{j\in[n]}$, but also need to take the capacities $\{q_j\}_{j\in[n]}$ into account. Informally, we will classify the buyers into three groups $[n] = L \cup H_S \cup H_B$:

- $L = \{j \in [n] : v_j < 1/k\}$. Because these group-$L$ distributions have *lower* bang-per-buck ratios $v_j < 1/k$, conceivably the total contribution by this group to the EAR benchmark shall be small. Indeed, we will prove a *constant* upper bound $\sum_{j\in L} v_j q_j = O(1)$.
- $H_S = \{j \in [n] : v_j \geq 1/k \text{ and } v_j q_j < 1/(2k)\}$. In other words, the group-H_S distributions have *high* enough bang-per-buck ratios $v_j \geq 1/k$ but *small* capacities, i.e., $v_j q_j < 1/(2k)$. It turns out that the total contribution by this group is also small, and we also will prove a *constant* upper bound $\sum_{j\in H_S} v_j q_j = O(1)$.
- $H_B = \{j \in [n] : v_j \geq 1/k \text{ and } v_j q_j \geq 1/(2k)\}$. That is, these group-$H_B$ distributions have *high* enough bang-per-buck ratios and *big* enough capacities. Therefore, this group should contribute the most to the EAR benchmark. Taking into account the feasibility constraint, $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in \mathbb{R}_{\geq 0}$, we will show $\sum_{j\in H_B} v_j q_j = O(\log k)$.

The actual grouping criteria in our proof are more complicated than the above ones, in order to handle other technical issues.

Finally, we notice that our grouping criteria borrow ideas from the "budget-feasible mechanism" literature [12, 24, 40], where the target is to design approximately optimal mechanisms for the Knapsack Problem under the incentive concerns. We hope that these ideas can find more applications to the "simple vs. optimal mechanism design" research topic.

1.3 Further Related Works

The revenue gaps among the mentioned mechanisms, Myerson Auction, Sequential Posted Pricing, Anonymous Reserve, and Anonymous Pricing, are extensively studied in the literature. Below we provide an overview of the previous results (mainly in the single-item setting and in the k-unit settings). As a supplement, the reader can refer to the surveys [16, 31, 34] and the textbook [27].

AR vs. AP. This revenue gap studies the relative power between the auction schemes and the pricing schemes, when the price discrimination is not allowed. The previously known results in the single-item case are shown in the next table.

i.i.d. regular	$e/(e-1) \approx 1.58$	[11, Thm 6] & [27, Thm 4.13]
i.i.d. general	$\pi^2/6 \approx 1.64$	[32, Thm 2]
asymmetric regular		

asymmetric general		

In the k-unit case, an asymptotically tight bound $1/(1 - k^k/(e^k k!)) \approx 1/(1 - 1/\sqrt{2\pi k})$ for i.i.d. regular buyers is shown in [41, Section 4.2] and [27, Section 4.5]. Our new results settle the remaining pieces of the puzzle – even though the i.i.d. assumption and/or the regularity assumption are removed, this revenue gap is still of order $1 + \Theta(1/\sqrt{k})$.

SPM vs. AP. This revenue gap studies the power of price discrimination in pricing schemes. We summarize the known results and our new results, in both the single-item case and the k-unit case.

single-item case		SPM vs. AP	OPT vs. AP
i.i.d. regular	$e/(e-1) \approx 1.58$	[11, Thm 6] & [27, Thm 4.13]	
i.i.d. general	$2 - 1/n$	[19, Thm 3]	[27, Thm 4.9]
asymmetric regular	constant $C^* \approx 2.62$	[32, Thm 1]	[30, Thm 1]
asymmetric general	n	[3, Prop 6.1]	

k-unit case		SPM vs. AP	OPT vs. AP
i.i.d. regular	$1/(1 - k^k/(e^k k!))$ ⋄	[19, Thm 1]	[41, Sec 4.2]
i.i.d. general	$2 - k/n$	[19, Thm 3]	[27, Sec 4.5]
asymmetric regular	$\Theta(\log k)$	this work	
asymmetric general	n	[3, Prop 6.1]	

⋄ this bound is just *asymptotically* tight

OPT vs. AP. This revenue gap is to illustrate that even the simplest mechanism, Anonymous Pricing, can approximate the optimal revenue in quite general settings. Actually, in each of the single-item/k-unit, i.i.d./asymmetric, regular/general settings, this ratio "coincedentally" is equal to the SPM vs. AP revenue gap, namely $\mathfrak{R}_{\mathsf{OPT/AP}} = \mathfrak{R}_{\mathsf{SPM/AP}}$.[4] (But the results respectively for $\mathfrak{R}_{\mathsf{OPT/AP}}$ and $\mathfrak{R}_{\mathsf{SPM/AP}}$ are credited to different works.) For brevity, we summarize the results on the both revenue gaps together in the above tables.

Instead of the regularity assumption, the stronger *monotone-hazard-rate* (MHR) distributional assumption is also very standard in the mechanism design literature. The previous works [22, 29] study the OPT vs. AP revenue gap in the single-item i.i.d. MHR setting.

OPT vs. AR. This ratio studies the power of price discrimination in the auction schemes. When the value distributions are i.i.d. and regular, Myerson Auction and Anonymous Reserve turn out to be identical [39]. The results beyond the i.i.d. regular case are given below.

single-item case		
i.i.d. general	$2 - 1/n$	[27, Thm 4.9]
asymmetric regular	LB ≈ 2.15	[28, Sec 5] & [32, Thm 3]
	UB $= C^* \approx 2.62$	[28, Sec 5] & [30, Thm 1]
asymmetric general	n	[3, Prop 6.1]

[4]The reader may wonder why the revenue gaps $\mathfrak{R}_{\mathsf{OPT/AP}}$ and $\mathfrak{R}_{\mathsf{SPM/AP}}$ are equal, in each of the single-item/k-unit, i.i.d./asymmetric, regular/general settings. This is because, in each of these settings, the worst-case instance $\{F_j^*\}_{j \in [n]}$ of the OPT vs. AP problem has a nice property: for each F_j^*, the corresponding *virtual-value distribution* is supported on the non-positive semiaxis $(-\infty, 0]$ plus a *single positive number* $v_j^* > 0$. When an instance satisfies this property, we can adopt the arguments in [32, Lemma 1] to show that OPT and SPM extract the same amount of revenue, which implies $\mathfrak{R}_{\mathsf{OPT/AP}} = \mathfrak{R}_{\mathsf{SPM/AP}}$.

k-unit case		
i.i.d. general	$2 - k/n$	[27, Sec 4.5]
asymmetric regular	$\Theta(\log k)$	this work
asymmetric general	n	[3, Prop 6.1]

Notably, the tight ratio in the single-item asymmetric regular setting is still unknown. Hartline and Roughgarden first prove that this ratio is between 2 and 4 [28, Section 5]. Afterwards, the lower bound is improved to ≈ 2.15 [32, Theorem 3]. But the best known upper bound just follows from the tight OPT vs. AP revenue gap $C^* \approx 2.62$ by implication. We highly believe this factor-C^* barrier can be broken, for which new techniques tailored for Anonymous Reserve rather than Anonymous Pricing are required.

Beyond the Anonymous Reserve mechanism, other simple auctions with the more powerful *personalized reserves* are also extensively studied [7, 28, 35].

OPT vs. SPM. This revenue gap investigates the relative power between the auction schemes and the pricing schemes, when the price discrimination is allowed. Indeed, the previous works [17, 26] show that this problem is identical to the *ordered prophet inequality* problem in stopping theory. In each of the single-item/k-unit i.i.d./asymmetric settings, the tight revenue gaps under/without the regularity assumption turn out to be the same (see, e.g., [41, Section 3.1]). The previous results in the single-item/k-unit cases are summarized below.

single-item case		
i.i.d.	constant $\beta \approx 1.34$	[15, Thm 1.3]
asymmetric	LB = $\beta \approx 1.34$	[15, Thm 1.3]
	UB = $1/(1 - 1/e + 1/27) \approx 1.49$	[18, Thm 1.1]

k-unit case		
i.i.d./asymmetric	LB = $1 + \Omega(1/\sqrt{k})$	[26, Thm 7]
	UB = $1 + O(1/\sqrt{k})$	[41, Sec 4.2] and [7, Sec 5].

Noticeably, the tight ratio in the single-item asymmetric setting is still unknown. The best known lower bound just follows from the tight "i.i.d." revenue gap $\beta \approx 1.34$ by implication. Recently, there is an outburst of activity on the upper bound [5, 7, 18], and the best known result is $1/(1 - 1/e + 1/27) \approx 1.49$ [18, Theorem 1.1]. It remains an interesting open question to further refine the upper bound.

Beyond the k-unit setting, the OPT vs. SPM revenue gap is also studied in the more general *matroid* setting. For this, the work [11, Theorem 5] first shows an upper bound of 2, and then [41, Section 4.1] improves it to $e/(e - 1) \approx 1.58$.

The Sequential Posted Pricing mechanism crucially leverages the order in which the buyers participate in the mechanism. Instead, the *order-oblivious* counterpart mechanisms are extensively studied as well [1, 2, 5, 7, 11, 18, 20].

Organization. In Section 2 we introduce the notation and requisite knowledge about the considered mechanisms. We investigate the Ex-Ante Relaxation vs. Anonymous Pricing problem in Section 3. For the Anonymous Reserve vs. Anonymous Pricing problem, a formal statement of our results and the proofs can be found from the full version of this paper.

2 NOTATION AND PRELIMINARIES

This section includes the notation to be adopted in this paper, and the basic knowledge about probability (e.g. the regular/triangle distributions) and the concerning mechanisms.

Notation. Denote by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{N}_{\geq 1}$) the set of all non-negative real numbers (resp. positive integers). For any pair of integers $b \geq a \geq 0$, define the sets $[a] \stackrel{\text{def}}{=} \{1, 2, \cdots, a\}$ and $[a : b] \stackrel{\text{def}}{=} \{a, a+1, \cdots, b\}$. Denote by $\mathbb{1}\{\cdot\}$ the indicator function. The function $|\cdot|_+$ maps a real number $z \in \mathbb{R}$ to $\max\{0, z\}$.

2.1 Probability

We use the bold letter $\mathbf{F} = \{F_j\}_{j \in [n]}$ to denote an instance (namely an n-dimensional *product distribution*), where F_j is the bid distribution of the buyer $j \in [n]$. For ease of notation, F_j also represents the corresponding cumulative density function (CDF).

We assume the CDF's $\{F_j\}_{j \in [n]}$ to be *left-continuous*, in the sense that when the j-th buyer has a random bid $b_j \sim F_j$ for a price-p item, his willing-to-pay probability is $\Pr[b_j \geq p]$ rather than $\Pr[b_j > p]$. We also define the inverse CDF $F_j^{-1}(y) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}_{\geq 0} : F_j(x) \geq y\}$ for any $y \in [0, 1]$; notice that possibly $F_j^{-1}(1) = \infty$. We say a distribution F_j *stochastically dominates* another $\overline{F}_j$, when $F_j(x) \leq \overline{F}_j(x)$ for all $x \in \mathbb{R}_{\geq 0}$. Further, an instance $\mathbf{F} = \{F_j\}_{j \in [n]}$ dominates another instance $\overline{\mathbf{F}} = \{\overline{F}_j\}_{j \in [n]}$, when F_j dominates $\overline{F}_j$ for each $j \in [n]$.

For a CDF F_j, we are also interested in two associated parameters (v_j, q_j). The *monopoly quantile* $q_j \in [0, 1]$ and the *monopoly price* $v_j \in \mathbb{R}_{\geq 0}$ are respectively given by

$$q_j \stackrel{\text{def}}{=} \operatorname*{arg\,max}_{q \in [0,1]} \{F_j^{-1}(1-q) \cdot q\} \qquad \text{and} \qquad v_j \stackrel{\text{def}}{=} F_j^{-1}(1-q_j).$$

If there are multiple maximizers q_j, we would choose the smallest q_j among the alternatives; notice that possibly $q_j = 0$ and $v_j = \infty$.

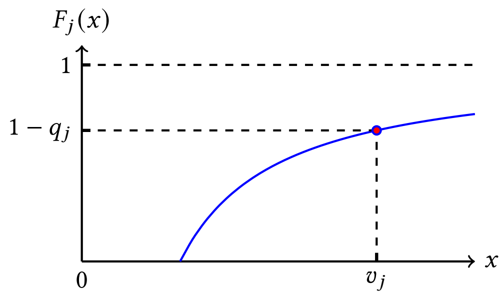
Sampling a bid profile from the instance $\mathbf{b} = (b_j)_{j \in [n]} \sim \mathbf{F}$, the i-th highest bids (for $i \in [n]$) $b_{(1)} \geq \cdots \geq b_{(i)} \geq \cdots \geq b_{(n)}$ will be of particular interest. We denote by D_i the corresponding distributions/CDF's, namely $D_i(x) = \Pr[b_{(i)} < x]$ for all $x \in \mathbb{R}_{\geq 0}$. Again, we assume $\{D_i\}_{i \in [n]}$ to be left-continuous. The formulas for the i-th highest CDF's are given below.

Fact 1 (Order Statistics). *For each $i \in [n+1]$, the i-th highest CDF is given by*

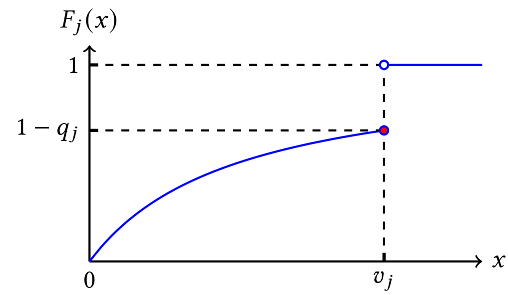
$$\begin{aligned} D_i(x) &= \sum_{t \in [0:i-1]} \sum_{|W|=t} \Big(\prod_{j \notin W} \Pr[b_j < x]\Big) \cdot \Big(\prod_{j \in W} \Pr[b_j \geq x]\Big) \\ &= \sum_{t \in [0:i-1]} \sum_{|W|=t} \Big(\prod_{j \notin W} F_j(x)\Big) \cdot \Big(\prod_{j \in W} (1 - F_j(x))\Big), \qquad \forall x \geq 0. \end{aligned}$$

Regular distribution. Denote by Reg this distribution family. According to [39], a distribution is regular $F_j \in$ Reg if and only if the *virtual value function* $\varphi_j(x) \stackrel{\text{def}}{=} x - \frac{1-F_j(x)}{f_j(x)}$ is non-decreasing on the support of F_j, where f_j is the probability density function (PDF). Such a regular CDF F_j is illustrated in Figure 3a.

Triangle distribution. This distribution family, denoted by Tri, is introduced in [3] and is a subset of the regular distribution family Reg. Such a distribution $\text{Tri}(v_j, q_j)$ is determined by the monopoly price $v_j \in \mathbb{R}_{\geq 0}$ and the monopoly quantile $q_j \in [0, 1]$. In precise, the corresponding CDF



(a) The CDF of a regular F_j



(b) The CDF of $\textsc{Tri}(v_j, q_j)$

Fig. 3. Demonstration for the regular distribution and the triangle distribution.

is given below and is illustrated in Figure 3b.

$$F_j(x) \overset{\text{def}}{=} \begin{cases} \frac{(1-q_j)\cdot x}{(1-q_j)\cdot x+v_j q_j}, & \forall x \in [0, v_j] \\ 1, & \forall x \in (v_j, \infty) \end{cases}.$$

2.2 Mechanisms

We focus on such a revenue maximization scenario: the seller has $k \in \mathbb{N}_{\geq 1}$ *homogeneous* items and faces $n \geq k$ *unit-demand* buyers, and the buyers draw their bids $\mathbf{b} = \{b_j\}_{j\in[n]} \sim \mathbf{F}$ independently from a publicly known product distribution $\mathbf{F} = \{F_j\}_{j\in[n]}$. For convenience, we interchange buyer/bidder.

In the bulk of the work, we will concern three mechanisms: Anonymous Pricing, Anonymous Reserve, and Ex-Ante Relaxation. Below we briefly introduce these mechanisms; for more details, the reader can refer to [27, Chapter 4].

Anonymous Pricing. In such a mechanism, the seller posts an *a priori* price $p \in \mathbb{R}_{\geq 0}$ to any single item; then in an arbitrary coming order, each of the first k coming buyers that are willing to pay the price $p \in \mathbb{R}_{\geq 0}$, will get an item by paying this price. Given any bid profile $\mathbf{b} \sim \mathbf{F}$, let $b_{(n+1)} \overset{\text{def}}{=} 0$ and reorder the bids such that $b_{(1)} \geq \cdots \geq b_{(i)} \geq \cdots \geq b_{(n+1)}$.

Depending on how many bids exceed the posted price, the mechanism gives a revenue of

$$\begin{aligned} \textsc{Rev}(\mathrm{AP}) &= \sum_{i\in[k]} i\cdot p\cdot \mathbb{1}\{b_{(i)} \geq p > b_{(i+1)}\} \;+\; k\cdot p\cdot \mathbb{1}\{b_{(k+1)} \geq p\} \\ &= \sum_{i\in[k]} p\cdot \mathbb{1}\{b_{(i)} \geq p\}. \end{aligned}$$

Taking the randomness over $\mathbf{b} \sim \mathbf{F}$ into account results in the expected revenue.

Fact 2 (Revenue Formula for Anonymous Pricing). *Under any posted price $p \in \mathbb{R}_{\geq 0}$, the* Anonymous Pricing *mechanism extracts an expected revenue of*

$$\mathrm{AP}(p, \mathbf{F}) \overset{def}{=} p\cdot \sum_{i\in[k]} (1 - D_i(p)).$$

Let $\mathrm{AP}(\mathbf{F}) \overset{def}{=} \max_{p\in\mathbb{R}_{\geq 0}}\{\mathrm{AP}(p, \mathbf{F})\}$ *denote the optimal* Anonymous Pricing *revenue.*

Anonymous Reserve. In such a mechanism, the seller sets an *a priori* reserve $r \in \mathbb{R}_{\geq 0}$ on any single item. When at most k bidders are willing to pay the reserve $r \in \mathbb{R}_{\geq 0}$, Anonymous Reserve has the same allocation/payment rule as Anonymous Pricing, thus the same revenue. But when at least $(k+1)$ bidders are willing to pay this reserve, each of the top-k bidders (with an arbitrary tie-breaking rule) wins an item by paying the $(k+1)$-th highest bid $b_{(k+1)} \geq r$.

Running on a specific bid profile $\mathbf{b} \sim \mathbf{F}$, the mechanism generates a revenue of

$$\textsc{Rev}(\mathsf{AR}) = \sum_{i \in [k]} i \cdot r \cdot \mathbb{1}\{b_{(i)} \geq r > b_{(i+1)}\} + k \cdot b_{(k+1)} \cdot \mathbb{1}\{b_{(k+1)} \geq r\}$$
$$= \sum_{i \in [k]} r \cdot \mathbb{1}\{b_{(i)} \geq r\} + k \cdot |b_{(k+1)} - r|_{+}.$$

Taking the randomness over $\mathbf{b} \sim \mathbf{F}$ into account gives the expected revenue. (Note that [10, Fact 1] get the revenue formula below in the single-item case $k = 1$.)

Fact 3 (Revenue Formula for Anonymous Reserve [10, Fact 1])**.** *Under any reserve $r \in \mathbb{R}_{\geq 0}$, the* Anonymous Reserve *mechanism extracts an expected revenue of*

$$\mathsf{AR}(r, \mathbf{F}) \stackrel{def}{=} r \cdot \sum_{i \in [k]} (1 - D_i(r)) + k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x.$$

Let $\mathsf{AR}(\mathbf{F}) \stackrel{def}{=} \max_{r \in \mathbb{R}_{\geq 0}} \{\mathsf{AR}(r, \mathbf{F})\}$ *denote the optimal* Anonymous Reserve *revenue.*

Ex-Ante Relaxation. This notion is introduced by [11]. Although just being a "fake" mechanism,[5] Ex-Ante Relaxation is useful to upper bound the revenue from the optimal truthful mechanism, Myerson Auction.

For a regular instance, an Ex-Ante Relaxation mechanism is specified by an allocation rule $\mathbf{q}' = \{q'_j\}_{j \in [n]} \in [0, 1]^n$. Here, each $q'_j \in [0, 1]$ represents the probability that the buyer $j \in [n]$ wins an item. This allocation rule is feasible iff $\sum_{j \in [n]} q'_j \leq k$, because we only have k items. The following fact characterizes the resulting "revenue".

Fact 4 (Revenue Formula for Ex-Ante Relaxation [11, Lemma 2])**.** *Given a regular instance* $\mathbf{F} = \{F_j\}_{j \in [n]}$*, under any feasible allocation rule* $\mathbf{q}' = \{q'_j\}_{j \in [n]} \in [0, 1]^n$ *that* $\sum_{j \in [n]} q'_j \leq k$*, the* Ex-Ante Relaxation *mechanism extracts an expected revenue of*

$$\mathsf{EAR}(\mathbf{q}', \mathbf{F}) \stackrel{def}{=} \sum_{j \in [n]} F_j^{-1}(1 - q'_j) \cdot q'_j$$

Remark 1. We will study the Ex-Ante Relaxation mechanism just for the regular instances. The revenue formulas for the irregular instances are more complicated, for which the reader can refer to [11, Lemma 2].

Revenue monotonicity. Based on the revenue formulas given in Facts 2 to 4, one can easily check the following fact (a.k.a. the revenue monotonicity in the literature).

Fact 5 (Revenue Monotonicity)**.** *Given that an instance* $\mathbf{F} = \{F_j\}_{j \in [n]}$ *stochastically dominates another instance* $\overline{\mathbf{F}} = \{\overline{F}_j\}_{j \in [n]}$*, the following hold:*

(1) $\mathsf{AP}(p, \mathbf{F}) \geq \mathsf{AP}(p, \overline{\mathbf{F}})$ *for any posted price* $p \in \mathbb{R}_{\geq 0}$*, and thus* $\mathsf{AP}(\mathbf{F}) \geq \mathsf{AP}(\overline{\mathbf{F}})$.
(2) $\mathsf{AR}(r, \mathbf{F}) \geq \mathsf{AR}(r, \overline{\mathbf{F}})$ *for any reserve* $r \in \mathbb{R}_{\geq 0}$*, and thus* $\mathsf{AR}(\mathbf{F}) \geq \mathsf{AR}(\overline{\mathbf{F}})$.
(3) $\mathsf{EAR}(\mathbf{q}', \mathbf{F}) \geq \mathsf{EAR}(\mathbf{q}', \overline{\mathbf{F}})$ *for any allocation* $\mathbf{q}' = \{q'_j\}_{j \in [n]} \in [0, 1]^n$ *with* $\sum_{j \in [n]} q'_j \leq k$.

[5] Namely, in the concerning Bayesian mechanism design setting, Ex-Ante Relaxation is unimplementable.

3 EX-ANTE RELAXATION VS. ANONYMOUS PRICING

In this section, we investigate the Ex-Ante Relaxation (EAR) vs. Anonymous Pricing (AP) problem, under the regularity assumption that $\mathbf{F} = \{F_j\}_{j \in [n]} \subseteq \text{Reg}$. Based on the revenue formulas (see Section 2.2), the revenue gap between both mechanisms is given by the optimal solution to the following mathematical program. Recall that D_i is the i-th highest bid distribution, and Reg is the family of all regular distributions.

$$\begin{aligned} \sup \quad & \mathsf{EAR}(\mathbf{q}', \mathbf{F}) \;=\; \sum_{j \in [n]} F_j^{-1}(1 - q'_j) \cdot q'_j && \text{(P2)} \\ \text{s.t.} \quad & \mathsf{AP}(p, \mathbf{F}) \;=\; p \cdot \sum_{i \in [k]} (1 - D_i(p)) \;\leq\; 1, && \forall p \in \mathbb{R}_{\geq 0}, \\ & \sum_{j \in [n]} q'_j \;\leq\; k, && \\ & \mathbf{q}' = \{q'_j\}_{j \in [n]} \in [0,1]^n, \;\; \mathbf{F} = \{F_j\}_{j \in [n]} \subseteq \text{Reg}, && \forall n \in \mathbb{N}_{\geq 1}. \end{aligned}$$

We will establish an $O(\log k)$ upper bound for the optimal solution to Program (P2), which is formalized as Theorem 3. Combine this result with the matching lower bound by [28, Example 5.4], then the revenue gap gets understood.

Theorem 3 (EAR vs. AP). *Given that the seller has $k \in \mathbb{N}_{\geq 1}$ homogeneous items and faces $n \geq k$ independent unit-demand buyers, who have regular value distributions $\mathbf{F} = \{F_j\}_{j \in [n]} \subseteq$ Reg, the revenue gap between* Ex-Ante Relaxation *and* Anonymous Pricing *is* $\mathfrak{R}_{\mathsf{EAR/AP}}(k) = \Theta(\log k)$.

Remark 2. We notice that under the stronger *monotone-hazard-rate* (MHR) distributional assumption, the EAR vs. AP revenue gap is still $\Theta(\log k)$. In particular, the $O(\log k)$ upper bound follows from Theorem 3 by implication, and the $\Omega(\log k)$ lower-bound instance by [28, Example 5.4] indeed satisfies the MHR condition.

We establish Theorem 3 in three steps. First, we give a reduction from a regular instance to a triangle instance, which preserves the feasibility; then we just need to optimize n pairs of monopoly price and quantile $\{(v_j, q_j)\}_{j \in [n]}$ instead of n regular distributions $\{F_j\}_{j \in [n]}$. Second, we relax the constraint $\mathsf{AP}(p, \mathbf{F}) \leq 1$ to a more tractable constraint, which avoids the correlation among the order statistics $\{D_i\}_{i \in [k]}$. Afterwards, we divide all buyers into three careful groups under certain criteria for $\{(v_j, q_j)\}_{j \in [n]}$, and separately bound the contribution from each group to the EAR revenue. The total EAR revenue turns out to be $O(\log k)$.

Reduction to triangle instances. For the single-item case $k = 1$, [3] show that the worst case of Program (P2) w.l.o.g. is achieved by a triangle instance. Indeed, their arguments work in the general case $k \in \mathbb{N}_{\geq 1}$ as well. Formally, we have the following lemma (see Figure 4 for a demonstration).

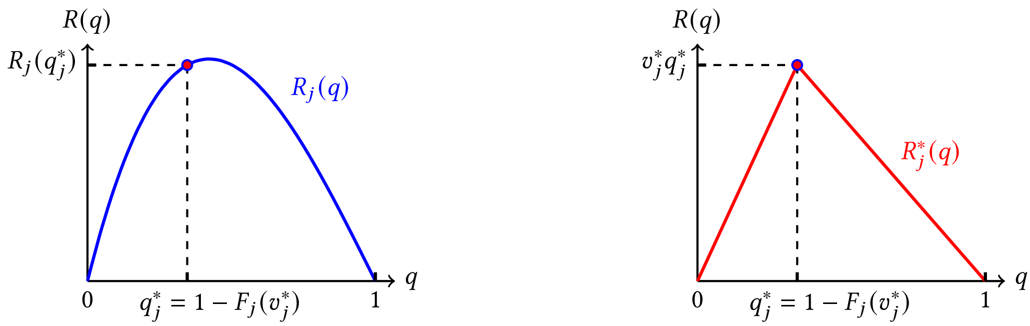
Lemma 1 (Reduction for EAR vs. AP [3, Lemma 4.1]). *Given a feasible solution* $(\mathbf{q}', \mathbf{F})$ *to Program* (P2)*, there exists another n-buyer feasible instance* $(\mathbf{q}^*, \mathbf{F}^*)$ *such that:*

(1) *The distributions $\mathbf{F}^* = \{F_j^*\}_{j \in [n]} \subseteq$ Tri are triangle distributions, and $\mathbf{q}^* = \{q_j^*\}_{j \in [n]} \in [0,1]^n$ (such that $\sum_{j \in [n]} q_j^* \leq k$) are the monopoly quantiles thereof.*

(2) *The* Ex-Ante Relaxation *revenue keeps the same, i.e.* $\mathsf{EAR}(\mathbf{q}^*, \mathbf{F}^*) = \mathsf{EAR}(\mathbf{q}', \mathbf{F})$.

(3) *The distributions $\mathbf{F}^* = \{F_j^*\}_{j \in [n]}$ are stochastically dominated by $\mathbf{F} = \{F_j\}_{j \in [n]}$ and thus, for any price $p \in \mathbb{R}_{\geq 0}$, the* Anonymous Pricing *revenue drops, i.e.* $\mathsf{AP}(p, \mathbf{F}^*) \leq \mathsf{AP}(p, \mathbf{F})$.

In view of Lemma 1, to establish Theorem 3 we can focus on Program (P3) in place of the previous Program (P2). For a triangle distribution $\text{Tri}(v_j, q_j)$, where $v_j = F_j^{-1}(1 - q_j) \geq 0$ is the monopoly



(a) A concave revenue-quantile curve

(b) A triangular revenue-quantile curve

Fig. 4. Demonstration for the reduction in Lemma 1, in terms of the *revenue-quantile curves*. For a distribution F_j, its revenue-quantile curve is given by $R_j(q) = q \cdot F_j^{-1}(1-q)$ for $q \in [0, 1]$. This distribution F_j is regular iff the R_j is a concave function (as Figure 4a suggests). And the revenue-quantile curve of a triangle distribution is basically a triangle (i.e., a 2-piecewise linear function, as Figure 4b suggests); in particular, the two base angles have the tangent values v_j^* and $v_j^* q_j^* / (1 - q_j^*)$, respectively.

price, we reuse F_j to denote its CDF. Recall Section 2.1 that $F_j(x) = \frac{(1-q_j)\cdot x}{(1-q_j)\cdot x + v_j q_j}$ for all $x \le v_j$ and $F_j(x) = 1$ for all $x > v_j$.

$$\sup \qquad \mathsf{EAR}(\mathbf{F}) = \sum_{j \in [n]} v_j q_j \tag{P3}$$

$$\text{s.t.} \qquad \mathsf{AP}(p, \mathbf{F}) = p \cdot \sum_{i \in [k]} (1 - D_i(p)) \le 1, \qquad \forall p \in \mathbb{R}_{\ge 0}, \tag{C2}$$

$$\sum_{j \in [n]} q_j \le k, \tag{C3}$$

$$\mathbf{F} = \{\textsc{Tri}(v_j, q_j)\}_{j \in [n]} \subseteq \textsc{Reg}, \qquad \forall n \in \mathbb{N}_{\ge 1}.$$

For a single triangle distribution $\textsc{Tri}(v_j, q_j)$, the optimal Anonymous Pricing revenue from it equals $\mathsf{AP}(\textsc{Tri}(v_j, q_j)) = v_j q_j$, which $\le \mathsf{AP}(\mathbf{F}) \le 1$ due to constraint (C2). We thus add one more constraint

$$v_j q_j \le 1, \qquad \forall j \in [n]. \tag{C4}$$

Relaxing constraint (C2)**.** Given Program (P3), both the objective function $\mathsf{EAR}(\mathbf{F})$ and constraint (C3) are easy to deal with. However, constraint (C2) is rather complicated, because it involves the correlated top-k bids $\{b_{(i)}\}_{i \in [k]}$ and the corresponding order CDF's $\{D_i\}_{i \in [k]}$ (as formulas of the individual CDF's $\{F_j\}_{j \in [n]}$) are cumbersome.

The following Lemma 2 relaxes constraint (C2) to another constraint. The resulting constraint is much easier to reason about. Namely, it avoids the correlation among the top-k bids $\{b_{(i)}\}_{i \in [k]}$ and admits a clean formula of the individual CDF's $\{F_j\}_{j \in [n]}$. Later we will see that after this relaxation, the optimal objective value of Program (P3) blows up just by a constant multiplicative factor. Denote $m \stackrel{\text{def}}{=} \lfloor \frac{k}{2} \rfloor \ge 2$ for convenience.

Lemma 2 (Relaxed Constraint)**.** *The following is a necessary condition for constraint* (C2)*:*

$$\sum_{j \in [n]} (1 - F_j(p)) \le \frac{4}{p}, \qquad \forall p \in \left[\frac{1}{m}, \frac{1}{2}\right].$$

PROOF OF LEMMA 2. Let us consider a specific price $p \in [\frac{1}{m}, \frac{1}{2}]$ for constraint (C2). For any $j \in [n]$, let the independent Bernoulli random variable $X_j \in \{0, 1\}$ denote whether the j-th buyer is willing to pay the price p, with the failure probability $\Pr[X_j = 0] = F_j(p)$. Then $X \stackrel{\text{def}}{=} \sum_{j \in [n]} X_j$ denotes how many buyers are willing to pay, and $Y \stackrel{\text{def}}{=} \min\{k, X\}$ denotes how many items are sold out in Anonymous Pricing.

We have the revenue $\mathrm{AP}(p, \mathbf{F}) = p \cdot \mathbf{E}[Y]$, and constraint (C2) is identical to $\mathbf{E}[Y] \le \frac{1}{p}$. For the equation given in Lemma 2, the LHS $= \sum_{j \in [n]} \Pr[X_j = 1] = \sum_{j \in [n]} \mathbf{E}[X_j] = \mathbf{E}[X]$.

On the opposite of Lemma 2, suppose that $\mathbf{E}[X] > \frac{4}{p}$. We have $\mathbf{E}[X] > 8$, given that the price $p \le \frac{1}{2}$. Since X is the sum of independent Bernoulli random variables, due to Chernoff bound, $\Pr[X < (1-\delta) \cdot \mathbf{E}[X]] < \frac{e^{-\delta \cdot \mathbf{E}[X]}}{(1-\delta)^{(1-\delta) \cdot \mathbf{E}[X]}}$ for any $\delta \in (0, 1)$. In particular,

$$\Pr\left[X < \frac{1}{2} \cdot \mathbf{E}[X]\right] \;<\; \left(\frac{2}{e}\right)^{\frac{1}{2} \cdot \mathbf{E}[X]} \;<\; \left(\frac{2}{e}\right)^{4} \;\approx\; 0.2931 \;<\; \frac{1}{2}, \tag{1}$$

where the first step follows by setting $\delta = \frac{1}{2}$; and the second step follows since $\mathbf{E}[X] > 8$.

And because $Y = \min\{k, X\}$, we further deduce that

$$\begin{aligned} \Pr\left[Y \ge \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\}\right] &= 1 - \Pr\left[Y < \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\}\right] \\ &= 1 - \Pr\left[X < \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\}\right] \\ &\ge 1 - \Pr\left[X < \frac{1}{2} \cdot \mathbf{E}[X]\right] \\ &> \frac{1}{2}, \end{aligned} \tag{2}$$

where the second step follows since $Y < \min\{k, \frac{1}{2} \cdot \mathbf{E}[X]\}$ holds only if $Y < k$, and thus only if $Y = X$; and the last step follows from Equation (1).

Based on the above arguments, we conclude a contradiction $\mathbf{E}[Y] > \frac{1}{p}$ as follows:

$$\begin{aligned} \mathbf{E}[Y] &\ge \Pr\left[Y \ge \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\}\right] \cdot \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\} \\ &> \frac{1}{2} \cdot \min\left\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\right\} \\ &\ge \frac{1}{2} \cdot \min\left\{k,\ \frac{2}{p}\right\} \\ &\ge \frac{1}{p}, \end{aligned}$$

where the second step applies Equation (2); the third step applies our assumption $\mathbf{E}[X] > \frac{4}{p}$; and the last step follows as $\frac{2}{p} \le 2m \le k$, given that $p \in [\frac{1}{m}, \frac{1}{2}]$ and $m = \lfloor \frac{k}{2} \rfloor$.

By refuting the assumption, we get $\mathbf{E}[X] \le \frac{4}{p}$ for any price $p \in [\frac{1}{m}, \frac{1}{2}]$. This completes the proof of Lemma 2. □

Given a triangle instance $\{\mathrm{TRI}(v_j, q_j)\}_{j \in [n]}$, by plugging the CDF formulas $\{F_j\}_{j \in [n]}$, we can reformulate Lemma 2 as follows:

$$\sum_{j \in [n] : v_j \ge p} \frac{v_j q_j}{(1 - q_j) \cdot p + v_j q_j} \;\le\; \frac{4}{p}, \qquad \forall p \in \left[\frac{1}{m},\ \frac{1}{2}\right]. \tag{C2′}$$

Grouping the buyers. To upper bound the objective function $\mathsf{EAR}(\mathbf{F}) = \sum_{j \in [n]} v_j q_j$, let us partition all the buyers into three groups $[n] = A \sqcup B \sqcup C$, where

$$A \stackrel{\text{def}}{=} \left\{ j \in [n] : v_j \geq \frac{1}{m} \text{ and } \frac{v_j q_j}{1 - q_j} \geq \frac{1}{m} \right\},$$
$$B \stackrel{\text{def}}{=} \left\{ j \in [n] : v_j \geq \frac{1}{m} \text{ and } \frac{v_j q_j}{1 - q_j} < \frac{1}{m} \right\},$$
$$C \stackrel{\text{def}}{=} \left\{ j \in [n] : v_j < \frac{1}{m} \right\}.$$

Regarding the groups A, B and C given above, their individual contributions to the benchmark $\mathsf{EAR}(\mathbf{F})$ actually admit the following bounds:

$$\sum_{j \in A} v_j q_j = O(\log k), \qquad \sum_{j \in B} v_j q_j \leq 8, \qquad \sum_{j \in C} v_j q_j \leq 3.$$

Suppose these bounds to be true, then combining them together immediately gives Theorem 3. Below we explain the intuitions of our grouping criteria (Remark 3), give an interesting observation for the instances that are constituted by "small" distributions (Remark 4), and then verify the above three bounds in the reverse order.

Remark 3 (Grouping Criteria). Recall the objective function of Program (P3), i.e., $\mathsf{EAR}(\mathbf{F}) = \sum_{j \in [n]} v_j q_j$, and constraint (C3), i.e., $\sum_{j \in [n]} q_j \leq k$. Here the monopoly revenues $\{v_j q_j\}_{j \in [n]}$ are the individual contributions by the triangle distributions $\{\mathrm{TRI}(v_j, q_j)\}_{j \in [n]}$, and (in the sense of the *Knapsack Problem*) the monopoly quantiles $\{q_j\}_{j \in [n]}$ can be regarded as the individual capacities. Thereby, the monopoly prices $\{v_j\}_{j \in [n]}$ somehow are the bang-per-buck ratios (i.e., the contribution to the EAR benchmark per unit of the capacity).

Of course we prefer those distributions with higher bang-per-buck ratios $\{v_j\}_{j \in [n]}$, but also need to take the capacities $\{q_j\}_{j \in [n]}$ into account. In particular:

- The group-C distributions have lower bang-per-buck ratios $v_j \leq 1/m$. So conceivably, the total contribution $\sum_{j \in C} v_j q_j$ by this group to the EAR benchmark shall be small, and we will prove an upper bound of 3.
- The group-B distributions have high enough bang-per-buck ratios $v_j \geq 1/m$ but small capacities, namely $v_j q_j / (1 - q_j) < 1/m$. It turns out that the total contribution $\sum_{j \in B} v_j q_j$ by this group is also small, and we will prove an upper bound of 8.
- The group-A distributions have high enough bang-per-buck ratios as well as big enough capacities. Thus, this group should contribute the most to the EAR benchmark, for which we will show $\sum_{j \in A} v_j q_j = O(\log k)$.

Indeed, our grouping criteria borrow ideas from the "budget-feasible mechanism design" literature [12, 24, 40], where the primary goal is to design approximately optimal mechanisms for the Knapsack Problem under the incentive concerns.

Remark 4 ("Small" Distributions). As argued in Section 1.2, regarding a *continuum of "small" buyers* (i.e., any single buyer has an infinitesimal contribution to the EAR benchmark, but there are infinitely many buyers $n \to \infty$), the EAR vs. AP revenue gap would be (at most) a universal constant for whatever $k \geq 1$. This is because every "small" buyer belongs to either group B or group C, and thus the EAR benchmark is at most $\sum_{j \in B \cup C} v_j q_j \leq 8 + 3 = 11$.

Revenue from group C**.** Since such a buyer $j \in C$ has a monopoly price $v_j < \frac{1}{m}$, we have

$$\sum_{j \in C} v_j q_j \leq \frac{1}{m} \cdot \sum_{j \in C} q_j \leq \frac{1}{m} \cdot \sum_{j \in [n]} q_j \leq \frac{1}{m} \cdot k \leq 3,$$

where the second step follows since $C \subseteq [n]$; the third step follows from constraint (C3); and the last step holds for $m = \lfloor \frac{k}{2} \rfloor$ and $k \geq 4$. (We will deal with the cases $k \in \{1, 2, 3\}$ separately, at the end of this section.)

Revenue from group B**.** Setting $p = \frac{1}{m}$ for constraint (C2′), we deduce that

$$\begin{aligned} 4m \;=\; \text{RHS of (C2}') \;\geq\; \text{LHS of (C2}') \;=\; & \sum_{j \in [n] : v_j \geq \frac{1}{m}} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + v_j q_j} \\ \geq\; & \sum_{j \in B} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + v_j q_j} \\ \geq\; & \sum_{j \in B} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + (1 - q_j) \cdot \frac{1}{m}} \\ \geq\; & \frac{m}{2} \cdot \sum_{j \in B} v_j q_j, \end{aligned}$$

where the second line follows since $\{j \in [n] : v_j \geq \frac{1}{m}\} \supseteq B$ (see the definition of B); the third line follows since $\frac{v_j q_j}{1 - q_j} < \frac{1}{m}$ for any $j \in B$; and the last line drops the $(1 - q_j)$ terms and then rearranges the formula.

Rearranging the above equation immediately gives $\sum_{j \in B} v_j q_j \leq 8$, as desired.

Revenue from group A**.** To verify the upper bound about this group, we shall generalize the definition of A, and get a chain of subgroups $A = A_m \supseteq A_{m-1} \supseteq \cdots \supseteq A_2$:

$$A_t \stackrel{\text{def}}{=} \left\{ j \in [n] : v_j \geq \frac{1}{t} \text{ and } \frac{v_j q_j}{1 - q_j} \geq \frac{1}{t} \right\}, \qquad \forall t \in [2 : m].$$

Given an index $t \in [2 : m]$, by setting $p = \frac{1}{t} \in [\frac{1}{m}, \frac{1}{2}]$ for constraint (C2′), we deduce that

$$\begin{aligned} 4t \;=\; \text{RHS of (C2}') \;\geq\; \text{LHS of (C2}') \;=\; & \sum_{j \in [n] : v_j \geq \frac{1}{t}} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{t} + v_j q_j} \\ \geq\; & \sum_{j \in A_t} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{t} + v_j q_j} \\ \geq\; & \sum_{j \in A_t} \frac{v_j q_j}{v_j q_j + v_j q_j} \\ =\; & \frac{1}{2} \cdot |A_t|, \end{aligned}$$

where the second step follows because $\{j \in [n] : v_j \geq \frac{1}{m}\} \supseteq A_t$ (see the definition of A_t); and the third step follows because $(1 - q_j) \cdot \frac{1}{t} \leq v_j q_j$ for each $j \in A_t$.

Based on the above equation, we easily bound the cardinality $|A_t| \leq 8t$ for each $t \in [2 : m]$. Combining the above arguments together gives

$$\begin{aligned} \sum_{j \in A} v_j q_j \;=\; \sum_{j \in A_m} v_j q_j \;=\; & \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \sum_{j \in A_t \setminus A_{t-1}} v_j q_j \\ \leq\; & \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \sum_{j \in A_t \setminus A_{t-1}} \frac{1}{t - 1} \cdot 1 \\ =\; & \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \frac{|A_t| - |A_{t-1}|}{t - 1} \end{aligned}$$

$$\begin{aligned}
&= \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + \frac{|A_m|}{m-1} + \sum_{t\in[3:m]} |A_t|\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&\le \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + \frac{8m}{m-1} + \sum_{t\in[3:m]} 8t\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&\le \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + 16 + \sum_{t\in[3:m]} 8t\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&= \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + 8 + \sum_{t\in[m-1]} \frac{8}{t}, \qquad (3)
\end{aligned}$$

where the second line follows because the monopoly price $v_j \in (\frac{1}{t-1}, \frac{1}{t}]$ for each $j \in A_t \setminus A_{t-1}$ (see the definitions of A_t and A_{t-1}), and the monopoly quantiles $q_j \in [0,1]$ are bounded; the fifth line applies the bounds $|A_t| \le 8t$ for each $t \in [2:m]$; the sixth line holds for $m = \lfloor \frac{k}{2} \rfloor$ and $k \ge 4$; and the last line is by elementary calculation.

Because $v_j q_j \le 1$ for all $j \in A_2$ (see constraint (C4)) and $|A_2| \le 16$, we can bound the first term in Equation (3): $\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2} \le |A_2| - \frac{|A_2|}{2} \le 8$. Plug this into Equation (3):

$$\sum_{j\in A} v_j q_j \;\le\; 16 + \sum_{t\in[m-1]} \frac{8}{t} \;=\; O(\log k),$$

where the last step holds for $m = \lfloor \frac{k}{2} \rfloor$.

Upper bound when $k \in \{1,2,3\}$. Clearly, the optimal value $\mathfrak{R}_{\mathsf{EAR/AP}}(k)$ of Program (P2), which involves $k \in \mathbb{N}_{\ge 1}$ items in both mechanisms, is at most the revenue gap between the k-item Ex-Ante Relaxation and the 1-item Anonymous Pricing. The later revenue gap is given by the next mathematical program.

$$\begin{aligned}
\sup \quad & \sum_{j\in[n]} F_j^{-1}(1-q_j')\cdot q_j' && \text{(P4)}\\
\text{s.t.} \quad & p\cdot(1-D_1(p)) \;\le\; 1, && \forall p \in \mathbb{R}_{\ge 0},\\
& \sum_{j\in[n]} q_j' \le k, &&\\
& \mathbf{q}' = \{q_j'\}_{j\in[n]} \in [0,1]^n,\ \mathbf{F} = \{F_j\}_{j\in[n]} \subseteq \textsc{Reg}, && \forall n \in \mathbb{N}_{\ge 1}.
\end{aligned}$$

The only difference between Program (P4) and the one in [3, Section 4] is the constraint $\sum_{j\in[n]} q_j' \le k$ (rather than ≤ 1). We can resolve Program (P4) by following the exactly same steps as in [3, Section 4]. By doing so, we will get

$$\mathfrak{R}_{\mathsf{EAR/AP}}(k) \;\le\; \text{optimal value of (P4)} \;=\; 1 + \mathcal{V}(Q^{-1}(k)),$$

where the functions $\mathcal{V}(p) \overset{\text{def}}{=} p \cdot \ln(\frac{p^2}{p^2-1})$ and $Q(p) \overset{\text{def}}{=} \ln(\frac{p^2}{p^2-1}) - \frac{1}{2}\cdot\sum_{t=1}^{\infty} t^{-2}\cdot p^{-2t}$. Then we can derive Theorem 3 in the case $k \in \{1,2,3\}$ via numeric calculation, as the next table shows.

k	1	2	3
$1+\mathcal{V}(Q^{-1}(k))$	≈ 2.7184	≈ 3.7897	≈ 4.8111

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