Counting Independent Sets and Colorings on Random Regular Bipartite Graphs

Chao Liao
Shanghai Jiao Tong University, China
chao.liao.95@gmail.com

Jiabao Lin
Shanghai University of Finance and Economics, China
lin.jiabao@mail.shufe.edu.cn

Pinyan Lu
Shanghai University of Finance and Economics, China
lu.pinyan@mail.shufe.edu.cn

Zhenyu Mao
Shanghai University of Finance and Economics, China
zhenyu.mao.17@gmail.com

—— **Abstract** ——

We give a fully polynomial-time approximation scheme (FPTAS) to count the number of independent sets on almost every Δ-regular bipartite graph if $\Delta \geq 53$. In the weighted case, for all sufficiently large integers Δ and weight parameters $\lambda = \widetilde{\Omega}\left(\frac{1}{\Delta}\right)$, we also obtain an FPTAS on almost every Δ-regular bipartite graph. Our technique is based on the recent work of Jenssen, Keevash and Perkins (SODA, 2019) and we also apply it to confirm an open question raised there: For all $q \geq 3$ and sufficiently large integers $\Delta = \Delta(q)$, there is an FPTAS to count the number of q-colorings on almost every Δ-regular bipartite graph.

1 Introduction

Counting independent sets on bipartite graphs (#BIS) plays a significant role in the field of approximate counting. A wide range of counting problems in the study of counting CSPs [14, 6, 15] and spin systems [19, 20, 17, 7], have been proved to be #BIS-equivalent or #BIS-hard under approximation-preserving reductions (AP-reductions) [13]. Despite its great importance, it is still unknown whether #BIS admits a fully polynomial-time approximation scheme (FPTAS) or it is as hard as counting the number of satisfying assignments of Boolean formulas (#SAT) under AP-reduction.

In this paper, we consider the problem of approximating #BIS (and its weighted version) on random regular bipartite graphs. Random regular bipartite graphs frequently appear in the analysis of hardness of counting independent sets [34, 12, 38, 39, 17]. Therefore, understanding the complexity of #BIS on such graphs is potentially useful for gaining

insights into the general case. Let $Z(G, \lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$ where $\mathcal{I}(G)$ is the set of all independent sets of a graph G and $\lambda > 0$ is the weight parameter. This function also arises in the study of the hardcore model of lattice gas systems in statistical mechanics. Hence we usually call $Z(G, \lambda)$ the partition function of the hardcore model with fugacity λ.

In the case where input graphs are allowed to be nonbipartite, the approximability for counting the number of independent sets (#IS) is well understood. Exploiting the correlation decay properties of $Z(G, \lambda)$, Weitz [41] presented an FPTAS for graphs of maximum degree Δ at fugacity $\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$. On the hardness side, Sly [38] proved that, unless $NP = RP$, there is a constant $\varepsilon = \varepsilon(\Delta)$ that no polynomial-time approximation scheme exists for $Z(G, \lambda)$ on graphs of maximum degree Δ at fugacity $\lambda_c(\Delta) < \lambda < \lambda_c(\Delta) + \varepsilon(\Delta)$. Later, this result was improved at any fugacity $\lambda > \lambda_c(\Delta)$ [39, 16]. In particular, these results state that if $\Delta \leq 5$, there is an FPTAS for #IS on graphs of maximum degree Δ, otherwise there is no efficient approximation algorithm unless $NP = RP$.

The situation is different on bipartite graphs. No NP-hardness result is known even on graphs with unbounded degree. Surprisingly, Liu and Lu [29] designed an FPTAS for #BIS which only requires one side of the vertex partition to be of maximum degree $\Delta \leq 5$. On the other hand, it is #BIS-hard to approximate $Z(G, \lambda)$ at fugacity $\lambda > \lambda_c(\Delta)$ on biparite graphs of maximum degree $\Delta \geq 3$ [7].

Recently, Helmuth, Perkins, and Regts [25] developed a new approach via the polymer model and gave efficient counting and sampling algorithms for the hardcore model at high fugacity on certain finite regions of the lattice $\mathbb{Z}^d$ and on the torus $(\mathbb{Z}/n\mathbb{Z})^d$. Their approach is based on a long line of work [36, 37, 28, 1, 2, 35]. Shortly after that, Jenssen, Keevash, and Perkins [26] designed an FPTAS for the hardcore model at high fugacity on bipartite expander graphs of bounded degree. And they further extended the result to random Δ-regular bipartite graphs with $\Delta \geq 3$ at fugacity $\lambda > (2e)^{250}$. This is the first efficient algorithm for the hardcore model at fugacity $\lambda > \lambda_c(\Delta)$ on random regular bipartite graphs. A natural question is, can we design FPTAS for lower fugacity and in particular the problem #BIS on random regular bipartite graphs? Indeed, we obtain such results. Let $\mathcal{G}_{n,\Delta}^{\text{bip}}$ denote the set of all Δ-regular bipartite graphs with n vertices on both sides.

▶ **Theorem 1.** *For $\Delta \geq 53$ and fugacity $\lambda \geq 1$, with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.*

▶ **Theorem 2.** *For all sufficiently large integers Δ and fugacity $\lambda = \widetilde{\Omega}\left(\frac{1}{\Delta}\right)$ [1], with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.*

For notational convenience, we use the term "on almost every Δ-regular bipartite graph" to denote that a property holds with high probability (tending to 1 as $n \to \infty$) for randomly chosen graphs from $\mathcal{G}_{n,\Delta}^{\text{bip}}$.

Counting proper q-colorings on a graph is another extensively studied problem in the field of approximate counting [27, 4, 5, 10, 23, 22, 33, 9, 24, 18, 11, 31, 21]. In general graphs, if the number of colors q is no more than the maximum degree Δ, there may not be any proper coloring over the graph. Therefore, approximate counting is studied in the range that $q \geq \Delta + 1$. It was conjectured that there is an FPTAS if $q \geq \Delta + 1$, but the current best result is $q \geq \alpha\Delta + 1$ with a constant α slightly below $\frac{11}{6}$ [40, 8]. The conjecture was only confirmed for the special case $\Delta = 3$ [30].

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[1] This means that $\lambda \geq (c_1 \log^{c_2} \Delta)/\Delta$ for some constants $c_1, c_2 > 0$.

On bipartite graphs, the situation is quite different. For any $q \geq 2$, we know that there always exist proper q-colorings for every bipartite graph. For any $q \geq 3$, it is shown to be #BIS-hard but unknown to be #BIS-equivalent [13]. Using a technique analogous to that for #BIS, we obtain an FPTAS to count the number of q-colorings on random Δ-regular bipartite graphs for all sufficiently large integers $\Delta = \Delta(q)$ for any $q \geq 3$.

▶ **Theorem 3.** *For $q \geq 3$ and $\Delta \geq 100\overline{q}^{10}$ where $\overline{q} = \lceil q/2 \rceil$, with high probability (tending to 1 as $n \to \infty$) for a graph chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\mathrm{bip}}$, there is an FPTAS to count the number of q-colorings.*

This result confirms a conjecture in [26].

Our Technique

The classical approach to designing approximate counting algorithms is random sampling via Markov chain Monte Carlo (MCMC). However, it is known that the Markov chains are slowly mixing on random bipartite graphs for both independent set and coloring if the degree Δ is not too small. Taking #BIS as an example, a typical independent set of a random regular bipartite graph of degree at least 6 is unbalanced: it either chooses most of its vertices from the left side or the right side. Thus, starting from an independent set with most vertices from the left side, a Markov chain is unlikely to reach an independent set with most of its vertices from the right side in polynomial time.

Even so, a recent beautiful work exactly makes use of the above separating property to design approximate counting algorithms [26]. By making the fugacity $\lambda > (2e)^{250}$ sufficiently large, they proved that largest contribution to the partition function comes from extremely unbalanced independent sets, those which occupy almost no vertices on one side and almost all vertices on the other side. In particular, for a bipartite graph $G = (\mathcal{L}, \mathcal{R}, E)$ with n vertices on both sides, they identified two independent sets $I = \mathcal{L}$ and $I = \mathcal{R}$ as ground states as they have the largest weight λ^n among all the independent sets. They proved that one only needs to sum up the weights of states which are close to one of the ground states, for no state is close to both ground states and the contribution from the states which are far away from both ground states is exponentially small.

However, the ground state idea cannot be directly applied to counting independent sets and counting colorings since each valid configuration is of the same weight. We extend the idea of ground states to ground clusters, which is not a single configuration but a family of configurations. For example, we identify two ground clusters for independent sets, those which are entirely chosen from vertices on the left side and those which are entirely chosen entirely from vertices on the right side. If a set of vertices is entirely chosen from vertices on one side, it is obviously an independent set. Thus each cluster contains 2^n different independent sets. Similarly, we want to prove that we can count the configurations which are close to one of the ground clusters and then add them up. For counting colorings, there are multiple ground clusters indexed by a subset of colors $\emptyset \subsetneq X \subsetneq [q]$: colorings which color $\mathcal{L}$ only with colors from X and color $\mathcal{R}$ only with colors from $[q] \setminus X$.

Unlike the ground states in [26], our ground clusters may overlap with each other and some configurations are close to more than one ground cluster. In addition to proving that the number of configurations which are far away from all ground clusters is exponentially small, we also need to prove that the number of double counted configurations is small.

After identifying ground states and with respect to a fixed ground state, Jenssen, Keevash, and Perkins [26] defined a polymer model representing deviations from the ground state and rewrote the original partition function as a polymer partition function. We follow this

idea and define a polymer model representing deviations from a ground cluster. However, deviation from a ground cluster is much subtler than deviation from a single ground state. For example, if we define polymer as connected components from the deviated vertices in the graph, we cannot recover the original partition function from the polymer partition function. We overcome this by defining polymer as connected components in the graph G^2, where an edge of G^2 corresponds to a path of length at most 2 in the original graph. Here, a compatible set of polymers also corresponds to a family of configurations in the original problem, while it corresponds to a single configuration in [26].

It is much more common in counting problems that most contribution is from a neighborhood of some clusters rather than a few isolated states. So, we believe that our development of the technique makes it suitable for a much broader family of problems.

Organization of the paper

In this 10-page version, we only prove Theorem 1 which already explains the key technique for proving Theorem 2 and Theorem 3. The complete proof (and the modifications necessary) for these two can be found in the full version. In Section 2 we review necessary definitions and facts. In Section 3 we prove Theorem 1, where the proof is divided into four parts. The first part deals with the property of the independent sets on certain graphs. The second part uses the polymer model to approximate the number of independent sets. The third part discusses how to approximate the partition function of the polymer model. The last part puts these things together.

Independent work

Towards the end of this project, we learned that the authors of [26] obtained similar results in their upcoming journal version submission.

2   Preliminaries

2.1   Independent sets and random regular bipartite graphs

All graphs considered in this paper are unweighted, undirected, with no loops but may have multiple edges. Let $G = (V, E)$ be a graph. We use $d_G(u, w)$ to denote the distance between two vertices u, w in the graph G. For $\emptyset \subsetneq U, W \subseteq V$, define $d_G(U, W) = \min_{u \in U, w \in W} d_G(u, w)$. Let $U \subseteq V$ be a nonempty set. Let $N_G(U) = \{v \in V : d_G(\{v\}, U) = 1\}$ to be the neighborhood of U and emphasize that $N_G(U) \cap U = \emptyset$. We use $G[U]$ to denote the induced subgraph of G on U. Let E^2 be the set of unordered pairs (u, v) such that $u \neq v$ and $d_G(u, v) \leq 2$. We define G^2 to be the graph (V, E^2). It is clear that if the maximum degree of G is at most Δ, then the maximum degree of G^2 is at most Δ^2. An independent set of the graph G is a subset $U \subseteq V$ such that $(u, w) \notin E$ for any $u, w \in U$. We use $\mathcal{I}(G)$ to denote the set of all independent sets of G. The weight of an independent set I is $\lambda^{|I|}$ where $\lambda > 0$ is a parameter called fugacity. We use $Z(G, \lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$ to denote the partition function of the graph G. Clearly, $Z(G, 1)$ is the number of indepndent sets of G.

For two positive real numbers a and b, we say a is an ε-relative approximation to b for some $\varepsilon > 0$ if $\exp(-\varepsilon)b \leq a \leq \exp(\varepsilon)b$, or equivalently $\exp(-\varepsilon)a \leq b \leq \exp(\varepsilon)a$. A fully polynomial-time approximation scheme (FPTAS) is an algorithm that for every $\varepsilon > 0$ outputs an ε-relative approximation to $Z(G)$ in time $(|G|/\varepsilon)^C$ for some constant $C > 0$, where $Z(G)$ is some quantity, like the number of independent sets, of graphs G that we would like to compute.

We use $G \sim \mathcal{G}_{n,\Delta}^{\mathrm{bip}}$ to denote sampling a Δ-regular bipartite graph G with n vertices on both sides uniformly at random. We say a Δ-regular bipartite graph $G = (\mathcal{L}, \mathcal{R}, E)$ with n vertices on both sides is an (α, β)-expander if for all subsets $U \subseteq \mathcal{L}$ or $U \subseteq \mathcal{R}$ with $|U| \leq \alpha n$, $|N(U)| \geq \beta |U|$. This property is called the expansion property of G. We use $\mathcal{G}_{\alpha,\beta}^{\Delta}$ to denote the set of all Δ-regular bipartite (α, β)-expanders. It is known that a random regular bipartite graph is an expander with high probability.

2.2 The polymer model

Let G be a graph and Ω be a finite set. A polymer $\gamma = (\overline{\gamma}, \omega_{\overline{\gamma}})$ consists of a support $\overline{\gamma}$ which is a connected subgraph of G and a mapping $\omega_{\overline{\gamma}}$ which assigns to each vertex in $\overline{\gamma}$ some value in Ω. We use $|\overline{\gamma}|$ to denote the number of vertices of $\overline{\gamma}$. There is also a weight function $w(\gamma, \cdot) : \mathbb{C} \to \mathbb{C}$ for each polymer γ. There can be many polymers defined on the graph G and we use $\Gamma^* = \Gamma^*(G)$ to denote the set of all polymers defined on it. However, at the moment we do not give a constructive definition of polymers. Such definitions are presented when they are needed, see Section 3.2. We say two polymers γ_1 and γ_2 are compatible if $d_G(\overline{\gamma_1}, \overline{\gamma_2}) > 1$ and we use $\gamma_1 \sim \gamma_2$ to denote that they are compatible. For a subset $\Gamma \subseteq \Gamma^*$ of polymers, it is compatible if any two different polymers in this set are compatible. We define $\mathcal{S}(\Gamma^*) = \{\Gamma \subseteq \Gamma^* : \Gamma \text{ is compatible}\}$ to be the collection of all compatible subsets of polymers. For $\Gamma \in \mathcal{S}(\Gamma^*)$, we also define $|\overline{\Gamma}|$ to be the number of vertices of the subgraph $\overline{\Gamma}$ and let $\omega_{\overline{\Gamma}}$ be a mapping which assigns each vertex $v \in \overline{\Gamma}$ the value that $\omega_{\overline{\gamma}}$ assigns to v where γ is the unique polymer whose support contains vertex v. We say (Γ^*, w) is a polymer model defined on the graph G and the partition function of this polymer model is $\Xi(G, z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^*)} \prod_{\gamma \in \Gamma} w(\gamma, z)$, where z is a complex variable and $\prod_{\gamma \in \emptyset} w(\gamma, z) = 1$ by convention. The following theorem states conditions that $\Xi(G, z)$ can be approximated efficiently.

▶ **Theorem 4** ([25], Theorem 2.2). *Fix Δ and let $\mathcal{G}$ be a set of graphs of degree at most Δ. Suppose:*

- *There is a constant C such that for all $G \in \mathcal{G}$, the degree of $\Xi(G, z)$ is at most $C|G|$.*
- *For all $G \in \mathcal{G}$ and $\gamma \in \Gamma^*(G)$, $w(\gamma, z) = a_\gamma z^{|\overline{\gamma}|}$ where $a_\gamma \neq 0$ can be computed in time $\exp(O(|\overline{\gamma}| + \log_2 |G|))$.*
- *For every connected subgraph G' of every $G \in \mathcal{G}$, we can list all polymers $\gamma \in \Gamma^*(G)$ with $\overline{\gamma} = G'$ in time $\exp(O(|G'|))$.*
- *There is a constant $R > 0$ such that for all $G \in \mathcal{G}$ and $z \in \mathbb{C}$ with $|z| < R$, $\Xi(G, z) \neq 0$.*

Then for every z with $|z| < R$, there is an FPTAS for $\Xi(G, z)$ for all $G \in \mathcal{G}$.

The following condition by Koteckỳ and Preiss (KP-condition) is useful to show that $\Xi(G, z)$ is zero-free in certain regions.

▶ **Lemma 5** ([28]). *Suppose there is a function $a : \Gamma^* \to \mathbb{R}_{>0}$ and for every $\gamma^* \in \Gamma^*$,*
$$\sum_{\gamma : \gamma \not\sim \gamma^*} e^{a(\gamma)} |w(\gamma, z)| \leq a(\gamma^*). \text{ Then } \Xi(G, z) \neq 0.$$

To verify the KP-condition, usually we need to enumerate polymers and the following lemma is useful to bound the number of enumerated polymers.

▶ **Lemma 6** ([3]). *For any graph $G = (V, E)$ with maximum degree Δ and $v \in V$, the number of connected induced subgraphs of order $k \geq 2$ containing v is at most $(e\Delta)^{k-1}/2$. As a corollary, the number of connected induced subgraphs of order $k \geq 1$ containing v is at most $(e\Delta)^{k-1}$.*

2.3   Some useful lemmas

Throughout this paper, we use $H(x)$ to denote the binary entropy function

$$H(x) = -x \log_2 x - (1-x) \log_2(1-x), \quad x \in (0,1).$$

Moreover, we extend this function to the interval $[0,1]$ by defining $H(0) = H(1) = 0$. This is reasonable since $\lim_{x \to 0^+} H(x) = \lim_{x \to 1^-} H(x) = 0$.

▶ **Lemma 7.** *It holds that $H(x) \leq 2\sqrt{x(1-x)} \leq 2\sqrt{x}$ for all $0 \leq x \leq 1$.*

▶ **Lemma 8** ([32, Lemma 10.2]). *Suppose that n is a positive integer and $k \in [0,1]$ is a number such that kn is an integer. Then $\frac{2^{H(k)n}}{n+1} \leq \binom{n}{kn} \leq 2^{H(k)n}$.*

▶ **Lemma 9.** *For $b > a > 0$, the function $f(\lambda) = \lambda^a / (\lambda + 1)^b$ is monotonically increasing on $[0, \frac{a}{b-a}]$ and monotonically decreasing on $[\frac{a}{b-a}, +\infty)$.*

3   Counting independent sets for $\lambda \geq 1$

Throughout this section, we consider integers $\Delta \geq 53$, fugacity $\lambda \geq 1$ and set parameters ζ, α, β to be $\zeta = 1.28, \alpha = \frac{2.9}{\Delta}, \beta = \frac{\Delta}{2.9\zeta}$.

▶ **Lemma 10.** *For $\Delta \geq 53$, $\lim_{n \to \infty} \Pr_{G \sim \mathcal{G}_{n,\Delta}^{\mathrm{bip}}} \left[G \in \mathcal{G}_{\alpha,\beta}^{\Delta} \right] = 1$.*

The reader can find the detailed proof of the lemma above in the full version of the paper.

In the rest of this section, whenever possible, we will simplify notations by omitting superscripts, subscripts and brackets with the symbols between (but this will not happen in the statement of lemmas and theorems). For example, $Z(G, \lambda)$ may be written as Z if G and λ are clear from context.

3.1   Approximating $Z(G, \lambda)$

For all $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha,\beta}^{\Delta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\lambda \geq 1$, let $\mathcal{I}_{\mathcal{X}}(G) = \{I \in \mathcal{I}(G) : |I \cap \mathcal{X}| < \alpha n\}$ and $Z_{\mathcal{X}}(G, \lambda) = \sum_{I \in \mathcal{I}_{\mathcal{X}}(G)} \lambda^{|I|}$. The main result in this part is that we can use $Z_{\mathcal{L}}(G, \lambda) + Z_{\mathcal{R}}(G, \lambda)$ to approximate $Z(G, \lambda)$.

▶ **Lemma 11.** *For $\Delta \geq 53$ and $\lambda \geq 1$, there are constants $C = C(\Delta) > 1$ and $N = N(\Delta)$ so that for all $G \in \mathcal{G}_{\alpha,\beta}^{\Delta}$ with $n > N$ vertices on both sides, $Z_{\mathcal{L}}(G, \lambda) + Z_{\mathcal{R}}(G, \lambda)$ is a C^{-n}-relative approximation to $Z(G, \lambda)$.*

Proof. Apply Lemma 12 and Lemma 13. ◀

▶ **Lemma 12.** *For $\Delta \geq 3$ and $\lambda \geq 1$, there are constants $C = C(\Delta) > 1$ and $N = N(\Delta)$ so that for all $G \in \mathcal{G}_{\alpha,\beta}^{\Delta}$ with $n > N$ vertices on both sides, $\sum_{I \in \mathcal{I}_{\mathcal{L}}(G) \cup \mathcal{I}_{\mathcal{R}}(G)} \lambda^{|I|}$ is a C^{-n}-relative approximation to $Z(G, \lambda)$.*

Proof. Let $\mathcal{B} = \mathcal{I} \setminus (\mathcal{I}_{\mathcal{L}} \cup \mathcal{I}_{\mathcal{R}})$. For any $I \in \mathcal{B}$, it follows from the definition of $\mathcal{B}$ that $|I \cap \mathcal{L}| \geq \alpha n$ and $|I \cap \mathcal{R}| \geq \alpha n$. Using the expansion property, we obtain $|N(I \cap \mathcal{L})| \geq \beta \lfloor \alpha n \rfloor$ and thus $|I \cap \mathcal{R}| \leq n - |N(I \cap \mathcal{L})| \leq (1 - 1/t)n$ where $1/t = \beta \lfloor \alpha n \rfloor / n \geq \alpha\beta - \beta/n$. Analogously, it holds that $|I \cap \mathcal{L}| \leq (1 - 1/t)n$. In the following, we assume $n \geq N_1$ for some $N_1 = N_1(\Delta) > 0$, such that $1 - 1/t \leq 1 - \alpha\beta + \beta/n = 1 - 1/\zeta + \beta/n \leq 0.219$. We obtain an upper bound of $\sum_{I \in \mathcal{B}} \lambda^{|I|}$ as follows:

(a) Consider an independent set $I \in \mathcal{B}$. Recall that $\alpha n \leq |I \cap \mathcal{L}| \leq (1 - 1/t)n$. We first enumerate a subset $U \subseteq \mathcal{L}$ with $\alpha n \leq |U| \leq (1-1/t)n$ and then enumerate all independent sets I with $I \cap \mathcal{L} = U$. Since $1 - 1/t < 1/2$, there are at most $n\binom{n}{\lfloor(1-1/t)n\rfloor} \leq n2^{H(1-1/t)n}$ ways to enumerate such a set U, where the inequality follows from Lemma 8.

(b) Now fix a set $U \subseteq \mathcal{L}$. Recall that every independent set $I \in \mathcal{B}$ satisfies $|I \cap \mathcal{R}| \leq (1-1/t)n$. Therefore $\sum_{I \in \mathcal{B}: |I \cap \mathcal{L}| = U} \lambda^{|I|} = \lambda^{|U|} \sum_{I \in \mathcal{B}: |I \cap \mathcal{L}| = U} \lambda^{|I \cap \mathcal{R}|} \leq \lambda^{(1-1/t)n} (\lambda + 1)^{(1-1/t)n}$.

(c) Combining the first two steps we obtain $\sum_{I \in \mathcal{B}} \lambda^{|I|} \leq n2^{H(1-1/t)n} \lambda^{(1-1/t)n}(\lambda+1)^{(1-1/t)n} = n2^{H(1-1/t)n}(\lambda^2 + \lambda)^{(1-1/t)n}$.

Using $\sum_{I \in \mathcal{I}_\mathcal{L} \cup \mathcal{I}_\mathcal{R}} \lambda^{|I|} \geq (\lambda + 1)^n$ and the upper bound above, we obtain

$$\frac{\sum_{I \in \mathcal{B}} \lambda^{|I|}}{\sum_{I \in \mathcal{I}_\mathcal{L} \cup \mathcal{I}_\mathcal{R}} \lambda^{|I|}} \leq \frac{n2^{H(1-1/t)n}(\lambda^2 + \lambda)^{(1-1/t)n}}{(\lambda + 1)^n} = n(f(\lambda))^n, \tag{1}$$

where $f(\lambda) = 2^{H(1-1/t)} \cdot \frac{\lambda^{1-1/t}}{(\lambda+1)^{1/t}}$. Since $1 - 1/t < 1/t$, it follows from Lemma 9 that $f(\lambda) \leq f(1) = 2^{H(1-1/t)-1/t} < 1$ for all $\lambda \geq 1$. So there exists some constant $C > 1$ such that Equation (1) $\leq n(f(1))^n < C^{-n}$ for all $n > N \geq N_1$ where $N = N(\Delta)$ is another sufficiently large constant. ◄

▶ **Lemma 13.** *For $\Delta \geq 53$ and $\lambda \geq 1$, there are constants $C > 1$ and N so that for all $G \in \mathcal{G}^\Delta_{\alpha,\beta}$ with $n > N$ vertices on both sides, $Z_\mathcal{L}(G, \lambda) + Z_\mathcal{R}(G, \lambda)$ is a C^{-n}-relative approximation to $\sum_{I \in \mathcal{I}_\mathcal{L}(G) \cup \mathcal{I}_\mathcal{R}(G)} \lambda^{|I|}$.*

Proof. For any $I \in \mathcal{I}_\mathcal{L} \cap \mathcal{I}_\mathcal{R}$, it holds that $|I \cap \mathcal{L}| < \alpha n$ and $|I \cap \mathcal{R}| < \alpha n$. Clearly $\sum_{I \in \mathcal{I}_\mathcal{L} \cup \mathcal{I}_\mathcal{R}} \lambda^{|I|} \geq (\lambda + 1)^n$. Therefore

$$\frac{\sum_{I \in \mathcal{I}_\mathcal{L} \cap \mathcal{I}_\mathcal{R}} \lambda^{|I|}}{\sum_{I \in \mathcal{I}_\mathcal{L} \cup \mathcal{I}_\mathcal{R}} \lambda^{|I|}} \leq (\lambda + 1)^{-n} \left(\sum_{k=0}^{\lfloor \alpha n \rfloor} \binom{n}{k} \lambda^k \right)^2 \leq n^2 \left(\frac{4^{H(\alpha)} \lambda^{2\alpha}}{\lambda + 1} \right)^n, \tag{2}$$

where the last inequality follows from Lemma 8. Recall that $\alpha = 2.9/\Delta$ and $\Delta \geq 53$. Then $\left. \frac{4^{H(\alpha)} \lambda^{2\alpha}}{\lambda + 1} \right|_{\lambda=1} \leq 0.76 < 1$. It follows from Lemma 9 that $4^{H(\alpha)} \lambda^{2\alpha}/(\lambda + 1)$ is monotonically decreasing in λ on $[1, \infty)$ for all fixed $\Delta \geq 53$. Thus Equation (2) $\leq \left(1/\left(0.76n^{2/n} \right) \right)^{-n} < C^{-n}$ for some constant $C > 1$ and for all $n > N$ where N is a sufficiently large constant. ◄

3.2 Approximating $Z_\mathcal{X}(G, \lambda)$

In this subsection, we discuss how to approximate $Z_\mathcal{X}(G, \lambda)$ for any graph $G \in \mathcal{G}^\Delta_{\alpha,\beta}, \mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\lambda \geq 1$. We will use the polymer model (see Section 2.2). First we constructively define the polymers we need. For any $I \in \mathcal{I}_\mathcal{X}(G)$, we can partition the graph $(G^2)[I \cap \mathcal{X}]$ into connected components $U_1, U_2, \ldots, U_k$ for some $k \geq 0$ (trivially $k = 0$ if $I \cap \mathcal{X} = \emptyset$). There are no edges in G^2 between U_i and U_j for any $1 \leq i \neq j \leq k$. If $k > 0$, let $p(I) = \{(U_1, \mathbf{1}_{U_1}), (U_2, \mathbf{1}_{U_2}), \ldots, (U_k, \mathbf{1}_{U_k})\}$ where $\mathbf{1}_{U_i}$ is the unique mapping from U_i to $\{1\}$. If $k = 0$, let $p(I) = \emptyset$. We define the set of all polymers to be $\Gamma^*_\mathcal{X}(G) = \bigcup_{I \in \mathcal{I}_\mathcal{X}(G)} p(I)$ and each element in this set is called a polymer. When the graph G and $\mathcal{X}$ are clear from the context, we simply denote by Γ^* the set of polymers. Clearly, p is a mapping from $\mathcal{I}_\mathcal{X}(G)$ to the set $\left\{ \Gamma \in \mathcal{S}(\Gamma^*_\mathcal{X}(G)) : |\overline{\Gamma}| < \alpha n \right\}$ since $\left| \overline{p(I)} \right| = |I \cap \mathcal{X}| < \alpha n$ for all $I \in \mathcal{I}_\mathcal{X}(G)$. For each polymer γ, define its weight function $w(\gamma, \cdot)$ as $w(\gamma, z) = \lambda^{|\overline{\gamma}|}(\lambda + 1)^{-|N(\overline{\gamma})|} z^{|\overline{\gamma}|}$, where z is a complex variable. The weight function can be computed in polynomial time in $|\overline{\gamma}|$. The partition function of the polymer model (Γ^*, w) on the graph G^2 is the following sum: $\Xi(z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^*)} \prod_{\gamma \in \Gamma} w(\gamma, z)$. Recall that two polymers γ_1 and γ_2 are compatible if $d_{G^2}(\overline{\gamma_1}, \overline{\gamma_2}) > 1$ and this condition is equivalent to $d_G(\overline{\gamma_1}, \overline{\gamma_2}) > 2$.

▶ **Lemma 14.** *For all bipartite graphs* $G = (\mathcal{L}, \mathcal{R}, E)$ *with* n *vertices on both sides,* $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ *and* $\lambda \geq 0$,

$$Z_{\mathcal{X}}(G, \lambda) = (\lambda + 1)^n \sum_{\Gamma \in \mathcal{S}(\Gamma_{\mathcal{X}}^*(G)) : |\overline{\Gamma}| < \alpha n} \prod_{\gamma \in \Gamma} w(\gamma, 1).$$

Proof. In the definition of polymers, p is a mapping from $\mathcal{I}_{\mathcal{X}}$ to $\{\Gamma \in \mathcal{S}(\Gamma^*) : |\overline{\Gamma}| < \alpha n\}$. Thus $Z_{\mathcal{X}}(G, \lambda) = \sum_{I \in \mathcal{I}_{\mathcal{X}}} \lambda^{|I|} = \sum_{\Gamma \in \mathcal{S}(\Gamma^*) : |\overline{\Gamma}| < \alpha n} \sum_{I \in \mathcal{I}_{\mathcal{X}} : p(I) = \Gamma} \lambda^{|I|}$. Fix $\Gamma \in \mathcal{S}(\Gamma^*)$ with $|\overline{\Gamma}| < \alpha n$. It holds that

$$\sum_{I \in \mathcal{I}_{\mathcal{X}} : p(I) = \Gamma} \lambda^{|I|} = \sum_{I \in \mathcal{I}_{\mathcal{X}} : I \cap \mathcal{X} = \overline{\Gamma}} \lambda^{|I|} = \lambda^{|\overline{\Gamma}|} (\lambda + 1)^{|(\mathcal{L} \sqcup \mathcal{R}) \setminus (\mathcal{X} \sqcup N_G(\overline{\Gamma}))|}, \tag{3}$$

where the last equality follows from $|\overline{\Gamma}| < \alpha n$. Since Γ is compatible, $N_G(\overline{\Gamma}) = \sqcup_{\gamma \in \Gamma} N_G(\overline{\gamma})$ and $|(\mathcal{L} \sqcup \mathcal{R}) \setminus (\mathcal{X} \sqcup N_G(\overline{\Gamma}))| = n - \sum_{\gamma \in \Gamma} |N_G(\overline{\gamma})|$. Thus Equation (3) $= \lambda^{\sum_{\gamma \in \Gamma} |\overline{\gamma}|} (\lambda + 1)^{n - \sum_{\gamma \in \Gamma} N(\overline{\gamma})} = (\lambda + 1)^n \prod_{\gamma \in \Gamma} \lambda^{|\overline{\gamma}|} (\lambda + 1)^{-|N(\overline{\gamma})|} = (\lambda + 1)^n \prod_{\gamma \in \Gamma} w(\gamma, 1)$. ◄

▶ **Lemma 15.** *For* $\Delta \geq 53$ *and* $\lambda \geq 1$, *there are constants* $C > 1$ *and* N *so that for all* $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha, \beta}^{\Delta}$ *with* $n > N$ *vertices on both sides and* $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$,

$$(\lambda + 1)^n \Xi(1) = (\lambda + 1)^n \sum_{\Gamma \in \mathcal{S}(\Gamma_{\mathcal{X}}^*(G))} \prod_{\gamma \in \Gamma} w(\gamma, 1)$$

is a C^{-n}-*relative approximation to* $Z_{\mathcal{X}}(G, \lambda)$.

Proof. It is clear that $Z_{\mathcal{X}}(G, \lambda) \geq (\lambda + 1)^n$. Then using Lemma 14 and Lemma 16 we obtain

$$\rho = \frac{(\lambda + 1)^n \Xi(1) - Z_{\mathcal{X}}(G, \lambda)}{Z_{\mathcal{X}}(G, \lambda)} \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^*) : |\overline{\Gamma}| \geq \alpha n} \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^*) : |\overline{\Gamma}| \geq \alpha n} 2^{-\beta |\overline{\Gamma}|}. \tag{4}$$

To enumerate each $\Gamma \in \mathcal{S}(\Gamma^*)$ with $|\overline{\Gamma}| \geq \alpha n$ at least once, we first enumerate an integer $\alpha n \leq k \leq n$, then since $\overline{\Gamma} \subseteq \mathcal{X}$, we choose k vertices from $\mathcal{X}$. Therefore, from Equation (4) we have

$$\rho \leq \sum_{k = \lceil \alpha n \rceil}^{n} \binom{n}{k} 2^{-\beta k} \leq \sum_{k = \lceil \alpha n \rceil}^{n} 2^{H(k/n)n} 2^{-\beta k} \leq \sum_{k = \lceil \alpha n \rceil}^{n} \left(2^{2\sqrt{n/k} - \beta} \right)^k \leq \sum_{k = \lceil \alpha n \rceil}^{n} \left(2^{2\sqrt{1/\alpha} - \beta} \right)^k,$$

where the inequalities follow from Lemma 8 and Lemma 7. Recall that $\zeta = 1.28$, $\alpha = 2.9/\Delta$, $\beta = \Delta/(2.9\zeta)$ and $\Delta \geq 53$. Let $f(\Delta) = 2\sqrt{1/\alpha} - \beta = 2\sqrt{\Delta/2.9} - \Delta/(2.9\zeta)$. We obtain $\rho \leq \frac{2^{f(\Delta)\alpha n}}{1 - 2^{f(\Delta)}} = \frac{\left(2^{2\sqrt{2.9/\Delta} - 1/\zeta} \right)^n}{1 - 2^{f(\Delta)}}$. Since $f(\Delta)$ is monotonically decreasing in Δ on $[53, +\infty)$, $\rho \leq \frac{\left(2^{2\sqrt{2.9/53} - 1/1.28} \right)^n}{1 - 2^{2\sqrt{53/2.9} - 53/(2.9 \times 1.28)}} \leq 0.81^n/0.98 < C^{-n}$ for some constant $C > 1$ and for all $n > N$ where N is a sufficiently large constant. ◄

▶ **Lemma 16.** *For all polymers* $\gamma \in \Gamma^*$ *defined by* $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha, \beta}^{\Delta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ *and* $\lambda \geq 1$, $|w(\gamma, z)| \leq (2^{-\beta} |z|)^{|\overline{\gamma}|}$. *As a corollary,* $w(\gamma, 1) \leq 2^{-\beta |\overline{\gamma}|}$ *and for all compatible* $\Gamma \subseteq \Gamma^*(G)$, $\prod_{\gamma \in \Gamma} w(\gamma, 1) \leq 2^{-\beta |\overline{\Gamma}|}$.

Proof. Let $n = |\mathcal{L}| = |\mathcal{R}|$ and let γ be any polymer. It follows from the definition of polymers that $|\overline{\gamma}| \leq \alpha n$ and by the expansion property, $|N(\overline{\gamma})| \geq \beta |\overline{\gamma}|$. Thus we have $|w(\gamma, z)| = \lambda^{|\overline{\gamma}|} (\lambda + 1)^{-|N(\overline{\gamma})|} |z|^{|\overline{\gamma}|} \leq (\lambda (\lambda + 1)^{-\beta})^{|\overline{\gamma}|} |z|^{|\overline{\gamma}|} \leq (2^{-\beta} |z|)^{|\overline{\gamma}|}$ where the last inequality follows from Lemma 9 since $\beta > 1$ and $\lambda \geq 1$. In particular, $w(\gamma, 1) \leq 2^{-\beta |\overline{\gamma}|}$. For any compatible Γ, it holds that $|\overline{\Gamma}| = \sum_{\gamma \in \Gamma} |\overline{\gamma}|$. Thus $\prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \prod_{\gamma \in \Gamma} 2^{-\beta |\overline{\gamma}|} = 2^{-\beta |\overline{\Gamma}|}$. ◄

3.3 Approximating the partition function of the polymer model

▶ **Lemma 17.** *For $\Delta \geq 53$ and $\lambda \geq 1$, there is an FPTAS for $\Xi(1)$ for all $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha,\beta}^{\Delta}$ and $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$.*

Proof. Apply the FPTAS in Theorem 4. ◀

To apply Theorem 4, we need to show that for the parameters in Lemma 17, the partition function has no zeros in the entire unit disk centered at 0.

▶ **Lemma 18.** *There is a constant $R > 1$ so that for $\Delta \geq 53$ and $\lambda \geq 1$, $\Xi(z) \neq 0$ for all $G \in \mathcal{G}_{\alpha,\beta}^{\Delta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $z \in \mathbb{C}$ with $|z| < R$.*

Proof. Set $R = 1.001$. For any $\gamma \in \Gamma^*$, let $a(\gamma) = t|\overline{\gamma}|$ where $t = \left(-1 + \sqrt{1 + 8e}\right)/(4e) \approx 0.346$. We will verify that the KP-condition $\sum_{\gamma:\gamma\not\sim\gamma^*} e^{t|\overline{\gamma}|}|w(\gamma,z)| \leq t|\overline{\gamma^*}|$ holds for any $\gamma^* \in \Gamma^*$ and any $|z| < R$. It then follows from Lemma 5 that $\Xi(z) \neq 0$ for any $|z| < R$. Recall that $d_{G^2}(\overline{\gamma}, \overline{\gamma^*}) \leq 1$ for all $\gamma \not\sim \gamma^*$. Thus there is always a vertex $v \in \overline{\gamma} \subseteq \mathcal{X}$ such that $v \in \overline{\gamma^*} \sqcup N_{G^2}(\overline{\gamma^*})$. The number of such vertices v is at most $\Delta^2|\overline{\gamma^*}|$. So to enumerate each $\gamma \not\sim \gamma^*$ at least once, we can: a) first enumerate a vertex v in $\mathcal{X} \cap \left(\overline{\gamma^*} \cup N_{G^2}(\overline{\gamma^*})\right)$; b) then enumerate an integer k from 1 to $\lfloor \alpha n \rfloor$; c) finally enumerate γ with $v \in \overline{\gamma}$ and $|\overline{\gamma}| = k$. Since $\overline{\gamma}$ is connected in G^2, applying Lemma 6 and using Lemma 16 to bound $|w(\gamma,z)|$ we obtain $\sum_{\gamma:\gamma\not\sim\gamma^*} e^{t|\overline{\gamma}|}|w(\gamma,z)| \leq \Delta^2|\overline{\gamma^*}| \left(e^t 2^{-\beta}|z| + \sum_{k=2}^{\lfloor \alpha n \rfloor} \left(e\Delta^2\right)^{k-1} 2^{-1} e^{tk} 2^{-\beta k}|z|^k\right)$. Let $x = e^{t+1}\Delta^2 2^{-\beta} R$. Since $|z| < R$, we obtain $\sum_{\gamma:\gamma\not\sim\gamma^*} e^{t|\overline{\gamma}|}|w(\gamma,z)| \leq \frac{x}{e}|\overline{\gamma^*}| \left(1 + \frac{1}{2}\sum_{k=2}^{\infty} x^{k-1}\right) = \frac{x(2-x)}{2e(1-x)} \cdot |\overline{\gamma^*}|$. Recall that $\zeta = 1.28$, $\beta = \Delta/(2.9\zeta)$ and $\Delta \geq 53$. Since $\Delta^2 2^{-\beta}$ is monotonically decreasing in Δ on $[53, +\infty)$, it holds that $x = e^{t+1}\Delta^2 2^{-\beta} R \leq \left(e^{t+1}\Delta^2 2^{-\beta} R\right)\big|_{\Delta=53} \leq 0.545$, and hence $\frac{x(2-x)}{2e(1-x)} < 0.33 < t$. ◀

3.4 Putting things together

Using the results from previous parts, we obtain our main result for counting independent sets.

▶ **Theorem 1.** *For $\Delta \geq 53$ and fugacity $\lambda \geq 1$, with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.*

Proof. This theorem follows from Lemma 10 and Lemma 19. ◀

▶ **Lemma 19.** *For $\Delta \geq 53$ and $\lambda \geq 1$, there is an FPTAS for $Z(G, \lambda)$ for all $G \in \mathcal{G}_{\alpha,\beta}^{\Delta}$.*

Proof. First we state our algorithm. See Algorithm 1 for a pseudocode description. The input is a graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha,\beta}^{\Delta}$ and an approximation parameter $\varepsilon > 0$. The output is a number $\widehat{Z}$ to approximate $Z(G, \lambda)$. We use $\Xi_{\mathcal{X}}(z)$ to denote the partition function of the polymer model $(\Gamma_{\mathcal{X}}^*(G), w)$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$. Let N_1, C_2, N_2, C_2 be the constants in Lemma 11 and Lemma 15, respectively. These two lemmas show that $(\lambda + 1)^n (\Xi_{\mathcal{L}}(1) + \Xi_{\mathcal{R}}(1))$ is a $C_1^{-n} + C_2^{-n} \leq 2\min(C_1, C_2)^{-n} \leq C^{-n}$-relative approximation to $Z(G, \lambda)$ for another constant $C > 1$ and all $n > N \geq \max(N_1, N_2)$ where N is another sufficiently large constant. If $n \leq N$ or $\varepsilon \leq 2C^{-n}$, we use the brute-force algorithm to compute $Z(G, \lambda)$. If $\varepsilon > 2C^{-n}$, we apply the FPTAS in Lemma 17 with approximation parameter $\varepsilon' = \varepsilon - C^{-n}$ to obtain outputs $\widehat{Z}_{\mathcal{L}}$ and $\widehat{Z}_{\mathcal{R}}$ which approximate $\Xi_{\mathcal{L}}(1)$ and $\Xi_{\mathcal{R}}(1)$, respectively. Let $\widehat{Z} = (\lambda + 1)^n (\widehat{Z}_{\mathcal{L}} + \widehat{Z}_{\mathcal{R}})$ be the output. It is clear that $\exp(-\varepsilon)\widehat{Z} \leq Z(G, \lambda) \leq \exp(\varepsilon)\widehat{Z}$.

■ **Algorithm 1** Counting independent sets at fugacity $\lambda \geq 1$ for $\Delta \geq 53$.

1: **Input:** *A graph* $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha,\beta}^{\Delta}$ *with* n *vertices on both sides and* $\varepsilon > 0$
2: **Output:** $\widehat{Z}$ *such that* $\exp(-\varepsilon)\widehat{Z} \leq Z(G, \lambda) \leq \exp(\varepsilon)\widehat{Z}$
3: **if** $n \leq N$ or $\varepsilon \leq 2C^{-n}$ **then**
4:     Use the brute-force algorithm to compute $\widehat{Z} \leftarrow Z(G, \lambda)$;
5:     Exit;
6: **end if**
7: $\varepsilon' \leftarrow \varepsilon - C^{-n}$;
8: Use the FPTAS in Lemma 17 to obtain $\widehat{Z}_{\mathcal{L}}$, an ε'-relative approximation to the partition function $\Xi(z)$ at $z = 1$ of the polymer model $(\Gamma_{\mathcal{L}}^{*}(G), w)$.
9: Use the FPTAS in Lemma 17 to obtain $\widehat{Z}_{\mathcal{R}}$, an ε'-relative approximation to the partition function $\Xi(z)$ at $z = 1$ of the polymer model $(\Gamma_{\mathcal{R}}^{*}(G), w)$.
10: $\widehat{Z} \leftarrow (\lambda + 1)^n \left(\widehat{Z}_{\mathcal{L}} + \widehat{Z}_{\mathcal{R}} \right)$;

Then we show that Algorithm 1 is indeed an FPTAS. It is required that the running time of our algorithm is bounded by $(n/\varepsilon)^{C_3}$ for some constant C_3 and for all $n > N_3$ where N_3 is a constant. Let $N_3 = N$. If $\varepsilon \leq 2C^{-n}$, the running time of the algorithm would be $2.1^n \leq (nC^n/2)^{C_3} \leq (n/\varepsilon)^{C_3}$ for sufficient large C_3. If $\varepsilon > 2C^{-n}$, the running time of the algorithm would be $(n/\varepsilon')^{C_4} = (n/(\varepsilon - C^{-n}))^{C_4} \leq (2n/\varepsilon)^{C_4} \leq (n/\varepsilon)^{C_3}$ for sufficient large C_3, where C_4 is a constant from the FPTAS in Lemma 17. ◄

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