Learning Reserve Prices in Second-Price Auctions

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Abstract

This paper proves the tight sample complexity of Second-Price Auction with Anonymous Reserve, up to a logarithmic factor, for each of all the value distribution families studied in the literature: [0, 1]-bounded, [1, H]-bounded, regular, and monotone hazard rate (MHR). Remarkably, the setting-specific tight sample complexity $\mathsf{poly}(\varepsilon^{-1})$ depends on the precision $\varepsilon \in (0, 1)$, but not on the number of bidders $n \geq 1$. Further, in the two bounded-support settings, our learning algorithm allows correlated value distributions.

In contrast, the tight sample complexity $\tilde{\Theta}(n) \cdot \mathsf{poly}(\varepsilon^{-1})$ of Myerson Auction proved by Guo, Huang and Zhang (STOC 2019) has a nearly-linear dependence on $n \geq 1$, and holds only for *independent* value distributions in every setting.

We follow a similar framework as the Guo-Huang-Zhang work, but replace their information theoretical arguments with a direct proof.

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1 Introduction

Bayesian auction theory assumes that the seller knows the prior value information of bidders and would design auctions/mechanisms by leveraging that information. In real-life applications, the priors are learned from historical data. How much data is needed to learn good auctions? This question motivates the research interest in the sample complexity for auction design, initiated by Cole and Roughgarden [24]. Concretely, it focuses on how many samples are needed, regarding the precision $\varepsilon \in (0,1)$ and the bidder population $n \in \mathbb{N}_{\geq 1}$, to

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¹ A very related topic, the sample complexity of optimal pricing for a single bidder, dates back to [28]. Also, some regret-minimization variants date earlier to [8, 12, 11].

Table 1 For Myerson Auction, the nearly-tight bounds in all settings are obtained by [35]. For Anonymous Reserve, the upper bounds in all settings follow from our Theorem 4, and the matching lower bounds are proved by [39, 35]. (In the MHR setting, the above lower bounds hold for *discrete* MHR distributions, but the best-known lower bounds for *continuous* MHR distributions are just $\widetilde{\Omega}(n \cdot \varepsilon^{-3/2})$ and $\widetilde{\Omega}(\varepsilon^{-3/2})$ [39, 35].)

	Myerson Auction		Anonymous Reserve		
[0,1]-additive	$\widetilde{\Theta}(n \cdot \varepsilon^{-2})$	- [35]	$\widetilde{\Theta}(\varepsilon^{-2})$	[39]	Thm 4
[1, H]	$\widetilde{\Theta}(n \cdot H \cdot \varepsilon^{-2})$		$\widetilde{\Theta}(H \cdot \varepsilon^{-2})$		
regular	$\widetilde{\Theta}(n \cdot \varepsilon^{-3})$		$\widetilde{\Theta}(\varepsilon^{-3})$		
MHR	$\widetilde{\Theta}(n \cdot \varepsilon^{-2})$		$\widetilde{\Theta}(\varepsilon^{-2})$	[35]	

learn an $(1 - \varepsilon)$ -approximate optimal auction. A long line of work had improved the sample complexity [24, 46, 27, 32, 52, 39]. The recent breakthrough result by Guo et al. [35] derived the tight sample complexity, up to poly-logarithmic factors, for all the value distribution families considered in the literature.

The above results all target the *revenue-optimal* single-item auction, namely the canonical Myerson Auction [48]. Nonetheless, Myerson Auction is fairly complicated and rarely used in real life [3]. In contrast, the AGT community has placed "simplicity" as a primary goal for auction design [37, 18, 1, 43, 42, 40]. In practice, one of the most popular auctions is Second-Price Auction with Anonymous Reserve (i.e., setting the same reserve price for every bidder), e.g., the auctions in eBay, AdX, and Adsense. We emphasize that such an auction has straightforward instructions:

Based on the bidders' value distributions, the seller carefully selects a reserve $r \in \mathbb{R}_{\geq 0}$ for the item. If all the bids are below r, then the seller retains the item. If only the highest bid reaches r, then the highest bidder wins the item by paying this reserve price. Otherwise (i.e., two or more bids reach r), the highest bidder wins the item by paying a price of the second-highest bid.

The significance and practicality of Anonymous Reserve naturally motivate a rich literature to study its approximability against Myerson Auction in terms of revenues [37, 36, 1, 43, 42, 41] and learnability [17, 46, 45, 51, 44].

The first "approximability" result was attained in Myerson's original paper [48]: When the value distributions are i.i.d. and satisfy the standard regularity assumption (see Section 2.1 for its definition), Myerson Auction reduces to Anonymous Reserve. Even if the distributional assumptions are greatly relaxed, as we quote from [37]: "In quite general settings, simple auctions like Anonymous Reserve provably approximates the optimal expected revenue, to within a small constant factor." Moreover, its learnability has been tackled in various contexts. For example, Cesa-Bianchi et al. [17] assumed i.i.d. and [0,1]-bounded value distributions and got a nearly optimal sample complexity of $\widetilde{\Theta}(\varepsilon^{-2})$.²

² Precisely, Cesa-Bianchi et al. studied the slightly different problem of regret minimization over a time horizon $t \in T$ (in terms of the cumulative revenue loss against the optimal Anonymous Reserve). While imposing a strong distributional assumption, the seller is assumed to know just the allocations and payments in the past rounds. Cesa-Bianchi et al. obtained a nearly optimal $\widetilde{\Theta}(\sqrt{T})$ -regret algorithm. This regret bound easily indicates the $\widetilde{\Theta}(\varepsilon^{-2})$ sample complexity bound.

Despite the above discussions, how many samples do we need to learn an $(1 - \varepsilon)$ -approximate optimal reserve price when the bidders have (possibly) distinct value distributions? This problem is essential to understand Anonymous Reserve but remains unsettled. Because we only need to learn a good reserve price, conceivably, the task should be much easier than learning Myerson Auction (which requires a complete understanding of all bidders' distributions). Our work shows that this is precisely the case: As Table 1 illustrates, Anonymous Reserve in comparison has dramatically smaller sample complexity. Remarkably, it depends only on the precision $\varepsilon \in (0,1)$ but not on the population $n \in \mathbb{N}_{\geq 1}$.

Our learning algorithm for Anonymous Reserve is clear and intuitive and thus may be more attractive in practice. First, we slightly "shrink" (in the sense of stochastic dominance) the empirical distributions determined by the samples, resulting in the dominated empirical distributions. Then, we compute the optimal reserve price for these dominated empirical distributions (or, when there are multiple optimal reserve prices, any of them). Employing this reserve price turns out to generate an $(1 - \varepsilon)$ -fraction as much revenue as the optimal Anonymous Reserve.

This framework was proposed by [35], and the analysis has two parts: revenue monotonicity and revenue smoothness. The revenue monotonicity of a specific auction means if a distribution instance \mathcal{F} stochastically dominates another \mathcal{F}' , then the two revenues satisfy that $\text{Rev}(\mathcal{F}) \geq \text{Rev}(\mathcal{F}')$. Since Myerson Auction and Anonymous Reserve both have this feature, for the analysis of revenue monotonicity, we can apply arguments à la [35]. Moreover, revenue smoothness means if two distribution instances are stochastically close (in some metric), then their revenues must also be close. Guo et al. establish the revenue smoothness of Myerson Auction via an elegant information theoretical argument. However, this proof scheme is inapplicable here, and instead, we will present a more direct proof.

Before elaborating on the new argument, let us briefly explain why Anonymous Reserve needs much fewer samples. The outcome of such an auction (i.e., the allocation and the payment) relies on the highest and second-highest bids, whose distributions suffice to determine the optimal reserve price. (In contrast, we must know the distributions of all bidders to implement Myerson Auction.) Since only two distributions rather than n distributions are involved, we can eliminate the dependence of the sample complexity on the population.

Nonetheless, the restriction on the highest and second-highest bids incurs another issue. In the model, we assume the bids to be *mutually independent*. This assumption is critical for the information theoretical arguments by Guo et al. and the optimality of Myerson Auction. Conversely, the highest two bids, in general, are *correlated*. It is highly nontrivial whether we can extend the information theoretical arguments to accommodate the correlated distributions.⁴ Thus, we prove the revenue smoothness by working directly with Anonymous Reserve revenue. The techniques derived here may find more applications in the future. (For example, they complement the extreme value theorems by [24, 14, 46].) We believe that a similar approach, associated with the tools by [42], can circumvent the information theoretical arguments by [35] and refine the poly-logarithmic factors in their sample complexity of Myerson Auction.

³ This reserve price must be *bounded* since the (dominated) empirical distributions determined by the samples are *bounded* almost surely (even in the *unbounded* regular/MHR settings).

⁴ The information theoretical arguments by [35] crucially rely on a particular form of *Pinsker's inequality*, which holds only for *independent* distributions. For Anonymous Reserve, in contrast, we need to deal with the generally *correlated* highest and second-highest distributions. So, we must abandon the proof scheme by Guo et al. and directly reason about Anonymous Reserve revenues.

Correlation. Another benefit of the direct arguments is that even if the bids are *arbitrarily correlated* (but capped with a specific high value), learning Anonymous Reserve needs the same amount of samples. This generalized model is arguably much more realistic. In this direction, an intriguing open problem is to study, given the correlated distributions, the sample complexity of the optimal mechanisms [29, 49] or the optima in certain families of *robust* mechanisms [50, 22, 10].

Data Compression. If we care about the *space complexity* of the learning algorithms, the improvement on Anonymous Reserve against Myerson Auction is even more significant. To learn Myerson Auction, we need $\widetilde{\mathcal{O}}(n^2) \cdot \operatorname{poly}(1/\varepsilon)$ space both to implement the algorithm and to store the output auction. (Note that each sample is an n-dimensional value vector.) Namely, we cannot predict the future bids and must record all details of the learned "virtual value functions". However, for Anonymous Reserve, since only the highest and second-highest bids are involved, we only need $\operatorname{poly}(1/\varepsilon)$ space to implement the algorithm and $\mathcal{O}(1)$ space to store the learned reserve price. This property is crucial to large markets, where historical data cannot be stored entirely in the memory, and we wish to handle it in very few passes (in the sense of streaming algorithms).

1.1 Comparison with Previous Approaches

To understand the sample complexity of Anonymous Reserve, an immediate attempt is to readopt the algorithm of [17], under minor modification to accommodate non-identical and even correlated value distributions (rather than just the i.i.d. ones). However, that algorithm crucially relies on a particular property of i.i.d. value distributions: we can infer \mathcal{F}_1 point-wise from \mathcal{F}_2 and vice versa, where \mathcal{F}_i denotes the CDF of the *i*-th highest bid. Without the i.i.d. assumption, the correlation between \mathcal{F}_1 and \mathcal{F}_2 is much more complex, which makes this attempt fail to work for our purpose.

Also, one may attempt the empirical revenue maximization scheme, which gives the nearly tight sample complexity for the similar task "optimal pricing $p_j \stackrel{\text{def}}{=} \operatorname{argmax}_p p \cdot (1 - F_j(p))$ for a single bidder F_j ". However, in the regular and the MHR settings, the proof of either sample complexity crucially relies on the underlying distributional assumption [28, 39]. For example, given a regular/MHR distribution F_j , either the optimal price p_j is unique, or all the optimal prices p_j form a connected interval. In contrast, given $n \in \mathbb{N}_{\geq 1}$ regular/MHR value distributions, \mathcal{F}_1 and \mathcal{F}_2 can have $\Omega(n)$ disconnected optimal prices $p_i \stackrel{\text{def}}{=} \operatorname{argmax}_p p \cdot (1 - \mathcal{F}_i(p))$ (see [43, Example 2]). Accordingly, \mathcal{F}_1 and \mathcal{F}_2 themselves cannot be regular/MHR, which rejects this attempt as well.

Another approach in the literature is to construct an ε -net of all candidate reserve prices, namely a $\operatorname{poly}(1/\varepsilon)$ -size hypothesis set \mathcal{H} , and figure out the best one in \mathcal{H} through the samples (see [27, 51, 32], which use this method to learn Myerson Auction). In fact, for the [0, 1]-bounded and [1, \mathcal{H}]-bounded settings, it is a folklore that ε -net type algorithms can attain the (nearly) tight sample complexity. However, the regular and MHR settings are less understood due to the lack of suitable tools, such as some particular extreme value theorems. Here we address this question; given the developed techniques, we can present such sample-optimal ε -net type algorithms in both settings.

However, we prefer the "shrink-then-optimize" framework of [35] for two reasons. First, ε -net type algorithms choose distinct hypothesis sets \mathcal{H} for different value distribution families, i.e., the distributional assumption somehow is part of the "input". By contrast, the new framework gives a *unified* and robust learning algorithm. In particular, different distributional assumptions induce different sample complexities but do not affect the algorithm

implementation. Second, our paper demonstrates that the new framework works not only for the input value distributions as in [35] but also for some "sketched" distributions, i.e., order statistics, for our purpose. It would be interesting to see further extensions of this framework.

1.2 Other Related Work

As mentioned, after the pioneering work of [24], the sample complexity of Myerson Auction had been improved in a sequence of papers [46, 27, 51, 32, 52] and was finally answered by [35]. En route, many techniques have been developed and may be helpful to mechanism design, learning theory, and information theory. For an outline of these techniques, the reader can turn to [35, Section 1].

Another related topic is the sample complexity of single-bidder revenue maximization. Now, the optimal mechanism is to post the monopoly price $p \stackrel{\text{def}}{=} \operatorname{argmax} \left\{ v \cdot \left(1 - F(v) \right) : v \in \mathbb{R}_{\geq 0} \right\}$ and then let the bidder make a take-it-or-leave-it decision. Again, the problem is self-contained only under one of the four assumptions in Table 1. Up to a poly-logarithmic factor, the optimal sample complexity is $\widetilde{\Theta}(\varepsilon^{-2})$ in the [0,1]-bounded additive-error setting [5, 39], $\widetilde{\Theta}(H \cdot \varepsilon^{-2})$ in the [1, H]-bounded setting [5, 39], $\widetilde{\Theta}(\varepsilon^{-3})$ in the continuous regular setting [28, 39], and $\widetilde{\Theta}(\varepsilon^{-2})$ in the MHR setting [35].

One can easily see that,⁵ in each of the four settings, the sample complexity of Anonymous Reserve must be lower bounded by the single-bidder sample complexity. Since each mentioned single-bidder lower bound matches with the claimed sample complexity of Anonymous Reserve in Table 1 (up to a logarithmic factor), it remains to establish the upper bounds in the bulk of this work.

To learn good posted prices for a single buyer, a complementary direction is to investigate how much expected revenue is achievable using exactly one sample. When the distribution is regular, [28] showed that using the sampled value as the price guarantees half of the optimal revenue. Indeed, this ratio is the best possible (in the sense of worst-case analysis) when the seller must post a deterministic price. However, better ratios are possible under certain adjustments to the model. First, if the seller can access the second sample, he can improve the ratio to 0.509 [4].⁶ Second, if a randomized price is allowed, the seller can get a better revenue guarantee by constructing a particular price distribution from the single sample [30]. Recently, [2] improved this ratio to 0.501, and proved that no randomized pricing scheme could achieve a 0.511-approximation. Moreover, if the buyer's distribution satisfies the stronger MHR condition, [39] gave a deterministic 0.589-approximation one-sample pricing scheme. Afterward, [2] improved this ratio to 0.644, and obtained a 0.648 impossibility result for any deterministic/randomized pricing scheme.

Another motivation of the "mechanism design via sampling" program is the recent research interest in *multi-item* mechanism design, where Myerson Auction or its naive generalizations are no longer optimal. The optimal multi-item mechanisms are often computationally/conceptually hard [25, 19, 20, 21]. Instead, a rich literature proves that *simple* multi-item mechanisms are learnable from polynomial samples and constantly approximate the optimal revenues [46, 6, 26, 47, 15, 52, 7, 33, 34].

⁵ E.g., imagine there is a dominant bidder in revenue maximization, and the other (n-1) bidders are negligible.

⁶ Concretely, [4] employs the empirical revenue maximization pricing scheme. That is, let $s_1 \ge s_2$ be the two samples, then choose s_1 as the posted price when $s_1 \ge 2 \cdot s_2$ and choose s_2 otherwise.

Organization. Notation and preliminaries are given below. In Section 3, we show our learning algorithm (see Algorithm 1) and present the analysis of revenue monotonicity. In Section 4, we present the analysis of revenue smoothness, hence the sample complexity promised in Table 1. In Section 5, we conclude this paper with a discussion on future research directions.

2 Notation and Preliminaries

Notation. Denote by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{N}_{\geq 1}$) the set of all non-negative real numbers (resp. positive integers). For any pair of integers $b \geq a \geq 0$, denote by [a] the set $\{1, 2, \dots, a\}$, and by [a:b] the set $\{a, a+1, \dots, b\}$. Denote by $\mathbb{I}(\cdot)$ the indicator function. The function $(\cdot)_+$ maps a real number $z \in \mathbb{R}$ to max $\{0, z\}$. For convenience, we interchange bid/value and bidder/buyer.

2.1 Probability

We use the calligraphic letter \mathcal{F} to denote an *input instance* (i.e., an *n*-dimensional *joint distribution*), from which the buyers $j \in [n]$ draw a value vector $\mathbf{s} = (s_j)_{j \in [n]} \in \mathbb{R}^n_{\geq 0}$. Particularly, if the value s_j 's are *independent* random variables (drawn from a *product distribution* \mathcal{F}), we further write $\mathcal{F} = \{F_j\}_{j \in [n]}$, where each F_j presents the marginal value distribution of the individual buyer $j \in [n]$. Regarding the Anonymous Reserve auctions (to be elaborated in Section 2.2), the highest and second-highest values \widehat{s}_1 and \widehat{s}_2 are of particular interest. We respectively denote by \mathcal{F}_1 and \mathcal{F}_2 the distributions of \widehat{s}_1 and \widehat{s}_2 .

As usual in the literature, we use the notations \mathcal{F} and \mathcal{F}_i (for $i \in \{1, 2\}$) and F_j (for $j \in [n]$) also to denote the corresponding CDF's. However, we assume that a single-dimensional CDF \mathcal{F}_i or F_j is left-continuous,⁷ in the sense that if a buyer has a random value $s \sim F$ for a price-p item, then his *unwilling-to-purchase probability* is $\mathbf{Pr}[s < p]$ rather than $\mathbf{Pr}[s \le p]$. Further, we say a distribution F stochastically dominates another distribution F' (or simply $F \succeq F'$) when their CDF's satisfy $F(v) \le F'(v)$ for any $v \in \mathbb{R}_{>0}$.

We investigate the input instance \mathcal{F} in four canonical settings. The first and second settings, where the support $\operatorname{supp}(\mathcal{F})$ is bounded within the *n*-dimensional hypercube $[0,1]^n$ or $[1,H]^n$ (for a given real number $H \geq 1$), are clear.

In the third setting, the input instance is a product distribution $\mathcal{F} = \{F_j\}_{j \in [n]}$, where each F_j is a continuous regular distribution.⁸ Denote by f_j the corresponding PDF. According to [48], the regularity means the virtual value function

$$\varphi_j(v) \stackrel{\text{def}}{=} v - \frac{1 - F_j(v)}{f_j(v)}$$

is monotone non-decreasing on the support $supp(F_j)$.

In the last setting, the input instance $\mathcal{F} = \{F_j\}_{j \in [n]}$ is also a product distribution, but each F_j now may be a discrete or continuous (or even mixture) distribution that has a monotone hazard rate (MHR). Let us specify the MHR condition [9] in the next paragraph.

MHR Distribution. A discrete MHR instance $\mathcal{F} = \{F_j\}_{j \in [n]}$ must be supported on a discrete set $\{k\Delta : k \in \mathbb{N}_{\geq 1}\}$ (as Figure 1a demonstrates), where $\Delta > 0$ is a given step-size. For each $j \in [n]$, consider the step function $G_j(v) \stackrel{\text{def}}{=} \ln (1 - F_j(v))$ (marked in blue) as well as the piece-wise linear function L_j (marked in gray) determined by the origin (0, 0) and the

⁷ For the *n*-dimensional input distribution \mathcal{F} , we never work with its CDF directly.

⁸ More precisely, F_j can have a unique probability mass at its support supremum.

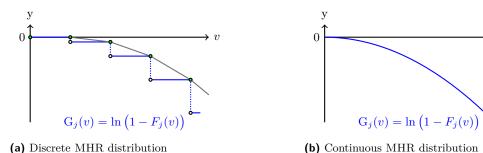


Figure 1 Demonstration for discrete and continuous MHR distributions.

"¬"-type points $(k \cdot \Delta, G_j(k \cdot \Delta))$'s (marked in green). The MHR condition holds iff each L_j is a concave function. Moreover, for a *continuous* MHR instance $\mathcal{F} = \{F_j\}_{j \in [n]}$, each individual F_j is supported on a possibly distinct interval. The MHR condition holds iff each $G_j(v) \stackrel{\text{def}}{=} \ln(1 - F_j(v))$ is a concave function on its own support, as Figure 1b illustrates.

2.2 Anonymous Reserve

In a Second-Price Auction with Anonymous Reserve, the seller posts an a priori reserve $r \in \mathbb{R}_{\geq 0}$ to the item. There are three possible outcomes: (i) when no buyer has a value of at least the reserve r, the auction would abort; (ii) when there is exactly one such buyer, he would pay the reserve r for winning the item; (iii) when there are two or more such buyers, the highest-value buyer (with arbitrary tie-breaking rule) would pay the second-highest value (i.e., a price of at least the reserve r) for winning the item.

We now formulate the expected revenue from the above mechanism [17, Fact 1]. Sample a random value vector $\mathbf{s} = (s_j)_{j \in [n]} \sim \mathcal{F}$ and then denote by (\hat{s}_1, \hat{s}_2) the highest and second-highest values. By simulating the mechanism, we have

(outcome revenue) =
$$r \cdot \mathbb{1}(\widehat{s}_1 \ge r > \widehat{s}_2) + \widehat{s}_2 \cdot \mathbb{1}(\widehat{s}_2 \ge r)$$

= $r \cdot \mathbb{1}(\widehat{s}_1 \ge r) + (\widehat{s}_2 - r)_+$
 \Rightarrow (expected revenue) = $r \cdot \mathbf{Pr}[\widehat{s}_1 \ge r] + \mathbf{E}[(\widehat{s}_2 - r)_+].$

In order to comprehend the expected revenue (denoted by $AR(r, \mathcal{F})$ for brevity), we need to know nothing (e.g. the correlation between \hat{s}_1 and \hat{s}_2) but the *marginal* CDF's \mathcal{F}_1 and \mathcal{F}_2 . So, we may write $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, namely the "union" of the highest and second-highest CDF's. Equipped with the new notations, let us formulate the expected revenue more explicitly.

▶ Proposition 1 (Revenue Formula [17]). Under any reserve $r \in \mathbb{R}_{\geq 0}$, the corresponding Anonymous Reserve auction extracts an expected revenue of

$$AR(r, \mathcal{F}) = r \cdot (1 - \mathcal{F}_1(r)) + \int_r^{\infty} (1 - \mathcal{F}_2(x)) \cdot dx.$$

When the reserve $r \in \mathbb{R}_{\geq 0}$ is selected optimally, namely $r_{\mathcal{F}} \stackrel{\text{def}}{=} \operatorname{argmax} \{ \mathsf{AR}(r, \mathcal{F}) : r \in \mathbb{R}_{\geq 0} \}$ (which might be infinity), we simply write $\mathsf{AR}(\mathcal{F}) = \mathsf{AR}(r_{\mathcal{F}}, \mathcal{F})$. Based on the revenue formula in Proposition 1, one can easily check the next Proposition 2 via elementary algebra.

- ▶ Proposition 2. The following holds for any pair of instances $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ and $\mathcal{F}' = \mathcal{F}'_1 \uplus \mathcal{F}'_2$ that admits the stochastic dominance $\mathcal{F}_1 \succeq \mathcal{F}'_1$ and $\mathcal{F}_2 \succeq \mathcal{F}'_2$:
- 1. $AR(r, \mathcal{F}) \geq AR(r, \mathcal{F}')$ for any reserve $r \in \mathbb{R}_{\geq 0}$.
- 2. $AR(\mathcal{F}) \geq AR(\mathcal{F}')$.

For ease of presentation, we also need the extra notations below, and the next Proposition 3 (see the full version for its proof) will often be invoked in our later proof.

- The parameter $\beta \stackrel{\text{def}}{=} \frac{\ln(8m/\delta)}{m}$, in which $m \in \mathbb{N}_{\geq 1}$ represents the sample complexity and $\delta \in (0,1)$ denotes the failing probability of a learning algorithm.
- The empirical instance $\mathcal{E} = \mathcal{E}_1 \uplus \mathcal{E}_2$ is given by a number of $m \in \mathbb{N}_{\geq 1}$ samples. Consider the i-th highest entry of every sample, then the i-th highest empirical distribution \mathcal{E}_i is exactly the uniform distribution supported on these i-th highest entries. Equivalently, \mathcal{E}_i is the uniform distribution induced by m samples from the i-th highest distribution \mathcal{F}_i .
- The shaded instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$: every *i*-th highest shaded CDF \mathcal{F}_i is defined as $\widetilde{\mathcal{F}}_i(v) \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{F}}(\mathcal{F}_i(v))$ for all value $v \in \mathbb{R}_{\geq 0}$, where the function

$$\mathsf{S}_{\mathcal{F}}(x) \stackrel{\text{def}}{=} \min \left\{ 1, \ x + \sqrt{8\beta \cdot x \cdot (1 - x)} + 7\beta \right\}, \qquad \forall x \in [0, 1].$$

The shaded empirical instance $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \uplus \widetilde{\mathcal{E}}_2$: every *i*-th highest shaded empirical CDF $\widetilde{\mathcal{E}}_i$ is defined as $\widetilde{\mathcal{E}}_i(v) \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{E}}(\mathcal{E}_i(v))$ for all value $v \in \mathbb{R}_{\geq 0}$, where the function

$$\mathsf{S}_{\mathcal{E}}(x) \stackrel{\text{def}}{=} \min \left\{ 1, \ x + \sqrt{2\beta \cdot x \cdot (1 - x)} + 4\beta \right\}, \qquad \forall x \in [0, 1]$$

▶ Proposition 3. Both of $S_{\mathcal{F}}(x)$ and $S_{\mathcal{F}}(x)$ are non-decreasing functions on interval $x \in [0,1]$.

Regarding Proposition 3, all the above instances are well defined. Without ambiguity, we may write $\widetilde{\mathcal{F}}_i \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{F}}(\mathcal{F}_i)$ and $\widetilde{\mathcal{E}}_i \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{E}}(\mathcal{E}_i)$. In the next section, we will show certain properties of/among them and the input instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$.

Empirical Algorithm

In this section, we first present our learning algorithm and formalize our main results (given respectively in Algorithm 1 and Theorem 4). Afterwards, we probe into the learned Anonymous Reserve auction via the revenue monotonicity (cf. Proposition 2). As a result, the learning problem will be converted into proving a certain property (parameterized by $\beta = \frac{\ln(8m/\delta)}{m}$, where m is the sample complexity) of the concerning instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$.

Algorithm 1 Empirical Algorithm.

Input: sample matrix $S = (s_t, j)_{m \times n}$, where each row $(s_t, j)_{j \in [n]}$ is a sample drawn from \mathcal{F} Output: an $(1-\varepsilon)$ -approximately optimal Anonymous Reserve auction for instance $\mathcal F$

- 1: **for all** $i \in \{1, 2\}$ **do**
- Let $\widehat{s}_i = (\widehat{s}_{t,\ i})_{t \in [m]}$ be the row-wise *i*-th highest entries of the sample matrix S// Namely, reorder rows $(s_t, j)_{j \in [n]}$ so that $s_{t, (1)} \ge \cdots \ge s_{t, (n)}$, then $\widehat{s}_{t, i} \stackrel{\text{def}}{=} s_{t, (i)}$ Let \mathcal{E}_i be the *i*-th highest empirical CDF induced by the *i*-th highest sample \widehat{s}_i
- Let $\widetilde{\mathcal{E}}_i \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{E}}(\mathcal{E}_i)$ be the shaded counterpart of the *i*-th highest empirical CDF \mathcal{E}_i
- 5: end for
- 6: **return** the optimal reserve $r_{\widetilde{\mathcal{E}}}$ for $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \uplus \widetilde{\mathcal{E}}_2$ (under any tie-breaking rule)
- ▶ Theorem 4. With (1δ) confidence, the reserve $r_{\widetilde{\mathcal{E}}} \in \mathbb{R}_{\geq 0}$ output by Algorithm 1 gives a nearly optimal Anonymous Reserve revenue $\mathsf{AR}(r_{\widetilde{\mathcal{E}}},\ \mathcal{F}) \geq \mathsf{AR}(\mathcal{F}) \varepsilon$, conditioned on
- 1. $m = \mathcal{O}(\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$ and the instance \mathcal{F} is supported on $[0,1]^n$. Alternatively, $AR(r_{\widetilde{\varepsilon}}, \mathcal{F}) \geq (1 - \varepsilon) \cdot AR(\mathcal{F})$, conditioned on
- 2. $m = \mathcal{O}(\varepsilon^{-2} \cdot H \cdot (\ln \varepsilon^{-1} + \ln H + \ln \delta^{-1}))$ and the instance \mathcal{F} is supported on $[1, H]^n$.
- 3. $m = \mathcal{O}(\varepsilon^{-3} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$ and the instance \mathcal{F} is regular.
- **4.** $m = \mathcal{O}(\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$ and the instance \mathcal{F} is MHR.

Analysis via Revenue Monotonicity. The following Lemma 5 (see the full version for its proof) suggests that (with high confidence) the empirical instance $\mathcal{E} = \mathcal{E}_1 \uplus \mathcal{E}_2$ is close to the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ in the Kolmogorov distance.

▶ **Lemma 5.** With $(1 - \delta)$ confidence, for both $i \in \{1, 2\}$, the following holds for the *i*-th highest CDF \mathcal{F}_i and its empirical counterpart \mathcal{E}_i : for any value $v \in \mathbb{R}_{>0}$,

$$\left| \mathcal{E}_i(v) - \mathcal{F}_i(v) \right| \le \sqrt{2\beta \cdot \mathcal{F}_i(v) \cdot \left(1 - \mathcal{F}_i(v)\right)} + \beta.$$

By construction $(\widetilde{\mathcal{E}}_i \stackrel{\text{def}}{=} \mathsf{S}_{\mathcal{E}}(\mathcal{E}_i)$ for both $i \in \{1, 2\}$), the shaded empirical instance $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \uplus \widetilde{\mathcal{E}}_2$ must be dominated by the empirical instance $\mathcal{E} = \mathcal{E}_1 \uplus \mathcal{E}_2$, thus likely being dominated by the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ as well (in view of Lemma 5).

Instead, let us consider the shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$ derived directly from the original instance \mathcal{F} via the other function $S_{\mathcal{F}}(\cdot)$, i.e., $\widetilde{\mathcal{F}}_i \stackrel{\text{def}}{=} S_{\mathcal{F}}(\mathcal{F}_i)$ for both $i \in \{1, 2\}$. Compared to the earlier function $S_{\mathcal{E}}(\cdot)$, the current function $S_{\mathcal{F}}(\cdot)$ distorts the input $x \in [0, 1]$ to a greater extent:

$$S_{\mathcal{F}}(x) = \min \left\{ 1, \ x + \sqrt{8\beta \cdot x \cdot (1 - x)} + 7\beta \right\}$$

$$\geq \min \left\{ 1, \ x + \sqrt{2\beta \cdot x \cdot (1 - x)} + 4\beta \right\} = S_{\mathcal{E}}(x),$$

where the inequality is strict when $S_{\mathcal{F}}(x) < 1$. Given these and in view of Lemma 5 (that the empirical instance \mathcal{E} is close to the original instance \mathcal{F}), the two shaded instances $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ are likely to admit the dominance $\widetilde{\mathcal{E}}_i \succeq \widetilde{\mathcal{F}}_i$ for both $i \in \{1, 2\}$.

These two propositions are formalized as Lemma 6 (see the full version for its proof):

- ▶ **Lemma 6.** In the case of Lemma 5, which happens with (1δ) confidence, for both $i \in \{1, 2\}$, the following holds for the empirical i-th highest CDF $\widetilde{\mathcal{E}}_i$:
- 1. $\widetilde{\mathcal{E}}_i(v) \geq \mathcal{F}_i(v)$ for any $v \in \mathbb{R}_{\geq 0}$, i.e., $\widetilde{\mathcal{E}}_i$ is dominated by the given i-th highest CDF \mathcal{F}_i .
- **2.** $\widetilde{\mathcal{E}}_i(v) \leq \widetilde{\mathcal{F}}_i(v)$ for any $v \in \mathbb{R}_{\geq 0}$, i.e., $\widetilde{\mathcal{E}}_i$ dominates the shaded i-th highest CDF $\widetilde{\mathcal{F}}_i$.

Using the reserve $r_{\widetilde{\mathcal{E}}}$ output by Algorithm 1, the corresponding Anonymous Reserve auction extracts an expected revenue of $\mathsf{AR}(r_{\widetilde{\mathcal{E}}},\ \mathcal{F})$ from the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$. Below, we give a lower bound of this revenue, which is more convenient for later analysis.

▶ Lemma 7. In the case of Lemma 5, which happens with $(1-\delta)$ confidence, from the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, the Anonymous Reserve with a reserve of $r_{\widetilde{\mathcal{E}}}$ generates a revenue better than the optimal Anonymous Reserve revenue from the shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$:

$$\mathsf{AR}(r_{\widetilde{c}},\ \mathcal{F}) \geq \mathsf{AR}(r_{\widetilde{c}},\ \widetilde{\mathcal{F}}) = \mathsf{AR}(\widetilde{\mathcal{F}}).$$

Proof. Due to Lemma 6 and Proposition 2 (i.e., the revenue monotonicity with respect to Anonymous Reserve):

$$\begin{split} \mathsf{AR}(r_{\widetilde{\mathcal{E}}},\ \mathcal{F}) &\geq \mathsf{AR}(r_{\widetilde{\mathcal{E}}},\ \widetilde{\mathcal{E}}) \\ &\geq \mathsf{AR}(r_{\widetilde{\mathcal{F}}},\ \widetilde{\mathcal{E}}) \\ &\geq \mathsf{AR}(r_{\widetilde{\mathcal{F}}},\ \widetilde{\mathcal{E}}) \\ &\geq \mathsf{AR}(r_{\widetilde{\mathcal{F}}},\ \widetilde{\mathcal{F}}) = \mathsf{AR}(\widetilde{\mathcal{F}}). \end{split} \tag{Part 1 of Lemma 6: dominance } \widetilde{\mathcal{F}}_i \succeq \widetilde{\mathcal{E}}_i) \\ &\geq \mathsf{AR}(r_{\widetilde{\mathcal{F}}},\ \widetilde{\mathcal{F}}) = \mathsf{AR}(\widetilde{\mathcal{F}}). \end{split} \tag{Part 2 of Lemma 6: dominance } \widetilde{\mathcal{E}}_i \succeq \widetilde{\mathcal{F}}_i) \end{split}$$

This completes the proof of Lemma 7.

Remarkably, the lower-bound revenue $\mathsf{AR}(\widetilde{\mathcal{F}})$ is irrelevant to Algorithm 1, since we directly construct the shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$ from the original instance \mathcal{F} via the function $\mathsf{S}_{\mathcal{F}}(\cdot)$ (parameterized by $\beta = \frac{\ln(8m/\delta)}{m}$, where m is the promised sample complexity). Based on the above discussions, Theorem 4 immediately follows if we have

$$\begin{split} \mathsf{AR}(\widetilde{\mathcal{F}}) &\geq \mathsf{AR}(\mathcal{F}) - \varepsilon \\ \mathsf{AR}(\widetilde{\mathcal{F}}) &\geq (1-\varepsilon) \cdot \mathsf{AR}(\mathcal{F}) \end{split} \qquad \text{(the } [0,1]\text{-bounded setting)} \end{split}$$

These two inequalities will be justified in Section 4.

4 Revenue Smoothness

In this section, we will bound the additive or multiplicative revenue gap between the shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$ and the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$. First of all, one can easily check the next Claim 8 via elementary algebra.

ightharpoonup Claim 8. The following holds for the parameter $\beta \stackrel{\text{def}}{=} \frac{\ln(8m/\delta)}{m}$:

1.
$$\beta \leq \frac{\varepsilon^2}{12}$$
 when $m \geq 36\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 3)$. ([0,1]-bounded setting)

2.
$$\beta \leq \frac{\varepsilon^2 \cdot H^{-1}}{48}$$
 when $m = 144\varepsilon^{-2} \cdot H \cdot (\ln \varepsilon^{-1} + \ln H + \ln \delta^{-1} + 4)$. ([1, H]-bounded setting)

3.
$$\beta \leq \frac{\varepsilon^3}{2880}$$
 when $m \geq 11520\varepsilon^{-3} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 4)$. (continuous regular setting)

4.
$$\beta \leq \frac{\varepsilon^2}{1870}$$
 when $m \geq 5610\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 5)$. (MHR setting)

4.1 [0,1]-Bounded Setting

Given the sample complexity $m = \mathcal{O}(\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$ promised in Part 1 of Theorem 4, we safely assume $m \geq 36\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 3)$. Consider the function $S_{\mathcal{F}}(\cdot)$: for $x \in [0, 1]$,

$$\begin{split} \mathbf{S}_{\mathcal{F}}(x) &= \min \left\{ 1, \quad x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta \right\} \\ &\leq x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta \\ &\leq x + \sqrt{2\beta} + 7\beta \qquad \qquad \text{(as } x \cdot (1-x) \leq \frac{1}{4} \text{)} \\ &\leq x + \frac{1}{\sqrt{6}} \cdot \varepsilon + \frac{7}{12} \cdot \varepsilon^2 \qquad \qquad \text{(Part 1 of Claim 8: } \beta \leq \frac{\varepsilon^2}{12} \text{)} \\ &\leq x + \varepsilon, \qquad \qquad \text{(as } \frac{1}{\sqrt{6}} + \frac{7}{12} \approx 0.9916 < 1 \text{)} \end{split}$$

which means that $\widetilde{\mathcal{F}}_i(v) \leq \mathcal{F}_i(v) + \varepsilon$ for all value $v \in [0, 1]$ and both $i \in \{1, 2\}$. Let $r_{\mathcal{F}} \in [0, 1]$ denote the optimal reserve for the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$. Thus,

$$\begin{split} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\widetilde{\mathcal{F}}) &\leq \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}) - \mathsf{AR}(r_{\mathcal{F}}, \ \widetilde{\mathcal{F}}) \qquad (r_{\mathcal{F}} \text{ may not be optimal for } \widetilde{\mathcal{F}}) \\ &= r_{\mathcal{F}} \cdot \left(\widetilde{\mathcal{F}}_1(r_{\mathcal{F}}) - \mathcal{F}_1(r_{\mathcal{F}})\right) + \int_{r_{\mathcal{F}}}^1 \left(\widetilde{\mathcal{F}}_2(x) - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\leq r_{\mathcal{F}} \cdot \varepsilon + \int_{r_{\mathcal{F}}}^1 \varepsilon \cdot \mathrm{d}x = \varepsilon. \end{split}$$

This concludes the proof in the setting with [0, 1]-bounded support.

⁹ Note that the interval of integration can be safely truncated to the support supremum of $s_u = 1$.

4.2 [1, H]-Bounded Setting

Given the sample complexity $m = \mathcal{O}(\varepsilon^{-2} \cdot H \cdot (\ln \varepsilon^{-1} + \ln H + \ln \delta^{-1}))$ promised in Part 2 of Theorem 4, we safely assume $m \ge 144\varepsilon^{-2} \cdot H \cdot (\ln \varepsilon^{-1} + \ln H + \ln \delta^{-1} + 4)$. To see this amount of samples is sufficient to learn a nearly optimal Anonymous Reserve, the next two facts will be useful.

 \triangleright Claim 9. From the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, the optimal Anonymous Reserve revenue $AR(\mathcal{F})$ is at least the support infimum of $s_l = 1$.

Proof. Obvious, e.g. the item always gets sold out under a reserve of 1.

ightharpoonup Claim 10. For the original instance $\mathcal{F}=\mathcal{F}_1 \uplus \mathcal{F}_2$, there is an optimal Anonymous Reserve auction having a reserve of $r_{\mathcal{F}} \in \left[1, \ \mathcal{F}_1^{-1}(\frac{H-1}{H})\right] \subseteq [1,H]$.

Proof. When there are multiple alternative optimal reserves $r_{\mathcal{F}}$'s, we would select the *smallest* one. Clearly, the bound $\mathcal{F}_1^{-1}(\frac{H-1}{H})$ is at least the support infimum of $s_l = 1 \leq \mathcal{F}_1^{-1}(0)$. Actually, employing the reserve of 1 guarantees as much revenue as employing another reserve $r \in (\mathcal{F}_1^{-1}(\frac{H-1}{H}), H]$: recall the Anonymous Reserve revenue formula,

$$\begin{split} \mathsf{AR}(1,\ \mathcal{F}) - \mathsf{AR}(r,\ \mathcal{F}) &= 1 \cdot \left(1 - \mathcal{F}_1(1)\right) - r \cdot \left(1 - \mathcal{F}_1(r)\right) + \int_1^r \left(1 - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\geq 1 \cdot \left(1 - \mathcal{F}_1(1)\right) - r \cdot \left(1 - \mathcal{F}_1(r)\right) \qquad (r > \mathcal{F}_1^{-1}(\frac{H-1}{H}) \geq 1) \\ &\geq 1 \cdot \left(1 - \mathcal{F}_1(1)\right) - H \cdot \left(1 - \frac{H-1}{H}\right) \qquad (r \leq H \ \& \ \mathcal{F}_1(r) > \frac{H-1}{H}) \\ &= 1 - 1 = 0. \qquad \qquad (\text{as } \mathcal{F}_1(1) = 0) \end{split}$$

That is, under our tie-breaking rule, any reserve $r \in (\mathcal{F}_1^{-1}(\frac{H-1}{H}), H]$ cannot be optimal, which completes the proof of Claim 9.

Define a parameter B $\stackrel{\text{def}}{=} \mathcal{F}_2^{-1}(\frac{H-1}{H}) \in [1,H]$. As shown in the former [0,1]-bounded setting, the function $\mathsf{S}_{\mathcal{F}}(x) \leq x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta$ for $x \in [0,1]$. We deduce that 10

$$\begin{aligned} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\widetilde{\mathcal{F}}) &\leq \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}) - \mathsf{AR}(r_{\mathcal{F}}, \ \widetilde{\mathcal{F}}) & (r_{\mathcal{F}} \ \mathsf{may not be optimal to} \ \widetilde{\mathcal{F}}) \\ &= r_{\mathcal{F}} \cdot \left(\widetilde{\mathcal{F}}_1(r_{\mathcal{F}}) - \mathcal{F}_1(r_{\mathcal{F}})\right) + \int_{r_{\mathcal{F}}}^H \left(\widetilde{\mathcal{F}}_2(x) - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\leq (\mathrm{First \ Term}) + (\mathrm{Second \ Term}) + (\mathrm{Third \ Term}) + 7\beta \cdot H, \end{aligned} \tag{1}$$

where

(First Term)
$$\stackrel{\text{def}}{=} r_{\mathcal{F}} \cdot \sqrt{8\beta \cdot \mathcal{F}_{1}(r_{\mathcal{F}}) \cdot \left(1 - \mathcal{F}_{1}(r_{\mathcal{F}})\right)}$$
.
(Second Term) $\stackrel{\text{def}}{=} \int_{r_{\mathcal{F}}}^{\max\{r_{\mathcal{F}}, B\}} \sqrt{8\beta \cdot \mathcal{F}_{2}(x) \cdot \left(1 - \mathcal{F}_{2}(x)\right)} \cdot dx$.
(Third Term) $\stackrel{\text{def}}{=} \int_{\max\{r_{\mathcal{F}}, B\}}^{H} \sqrt{8\beta \cdot \mathcal{F}_{2}(x) \cdot \left(1 - \mathcal{F}_{2}(x)\right)} \cdot dx$.

We measure these terms in the next two lemmas.

▶ Lemma 11. (First Term) + (Second Term) $\leq \sqrt{8\beta \cdot H} \cdot \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}) = \sqrt{8\beta \cdot H} \cdot \mathsf{AR}(\mathcal{F}).$

Proof. Recall Claim 10 that $r_{\mathcal{F}} \leq \mathcal{F}_1^{-1}(\frac{H-1}{H})$, which implies $\mathcal{F}_1(r_{\mathcal{F}}) \leq \frac{H-1}{H}$ and thus $\frac{\mathcal{F}_1(r_{\mathcal{F}})}{1-\mathcal{F}_1(r_{\mathcal{F}})} \leq H-1 \leq H$. Consequently,

(First Term) =
$$\sqrt{8\beta \cdot \frac{\mathcal{F}_1(r_{\mathcal{F}})}{1 - \mathcal{F}_1(r_{\mathcal{F}})}} \cdot r_{\mathcal{F}} \cdot (1 - \mathcal{F}_1(r_{\mathcal{F}}))$$

¹⁰ Note that the interval of integration can be safely truncated to the support supremum of $s_u = H$.

$$\leq \sqrt{8\beta \cdot H} \cdot r_{\mathcal{F}} \cdot (1 - \mathcal{F}_1(r_{\mathcal{F}})).$$

Similarly, $\sqrt{\mathcal{F}_2(v) \cdot \left(1 - \mathcal{F}_2(v)\right)} \le \sqrt{H} \cdot \left(1 - \mathcal{F}_2(v)\right)$ whenever $v \le B = \mathcal{F}_2^{-1}(\frac{H-1}{H})$. Hence, ¹¹

(Second Term) =
$$\int_{r_{\mathcal{F}}}^{\max\{r_{\mathcal{F}}, B\}} \sqrt{8\beta \cdot \mathcal{F}_{2}(x) \cdot (1 - \mathcal{F}_{2}(x))} \cdot dx$$
$$\leq \sqrt{8\beta \cdot H} \cdot \int_{r_{\mathcal{F}}}^{\max\{r_{\mathcal{F}}, B\}} (1 - \mathcal{F}_{2}(x)) \cdot dx$$
$$\leq \sqrt{8\beta \cdot H} \cdot \int_{r_{\mathcal{F}}}^{H} (1 - \mathcal{F}_{2}(x)) \cdot dx.$$

Combining the above two inequalities together completes the proof of Lemma 11.

▶ Lemma 12. (Third Term) $\leq \sqrt{8\beta \cdot H}$.

Proof. Clearly, the second-highest CDF $\mathcal{F}_2(v) \leq 1$ for any value $v \in \mathbb{R}_{\geq 0}$. For any value $v \geq B = \mathcal{F}_2^{-1}(\frac{H-1}{H}) \in [1, H]$, we have $1 - \mathcal{F}_2(v) \leq \frac{1}{H}$. Accordingly,

(Third Term) =
$$\int_{\max\{r_{\mathcal{F}, B}\}}^{H} \sqrt{8\beta \cdot \mathcal{F}_{2}(x) \cdot \left(1 - \mathcal{F}_{2}(x)\right)} \cdot dx$$
$$\leq \int_{\max\{r_{\mathcal{F}, B}\}}^{H} \sqrt{8\beta/H} \cdot dx \leq H \cdot \sqrt{8\beta/H} = \sqrt{8\beta \cdot H}.$$

This completes the proof of Lemma 12.

Applying Lemmas 11 and 12 to inequality (1), we conclude that $AR(\widetilde{\mathcal{F}}) \geq (1 - \varepsilon) \cdot AR(\mathcal{F})$:

$$\begin{split} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\widetilde{\mathcal{F}}) &\leq \sqrt{8\beta \cdot H} \cdot \mathsf{AR}(\mathcal{F}) + \sqrt{8\beta \cdot H} + 7\beta \cdot H \\ &\leq (2 \cdot \sqrt{8\beta \cdot H} + 7\beta \cdot H) \cdot \mathsf{AR}(\mathcal{F}) \ \ (\mathsf{Claim} \ 9 \colon \ \mathsf{AR}(\mathcal{F}) \geq 1) \\ &\leq (\frac{\sqrt{6}}{3} \cdot \varepsilon + \frac{7}{48} \cdot \varepsilon^2) \cdot \mathsf{AR}(\mathcal{F}) \\ &\leq \varepsilon \cdot \mathsf{AR}(\mathcal{F}). \\ \end{split} \tag{Claim 8, Part 2: } \beta \geq \frac{\varepsilon^2 \cdot H^{-1}}{48}) \\ &\leq \varepsilon \cdot \mathsf{AR}(\mathcal{F}). \end{split}$$

This completes the proof in the [1, H]-bounded setting.

4.3 Continuous Regular Setting

Throughout this subsection, we assume that each buyer $j \in [n]$ independently draws his value (for the item) from a continuous regular distribution F_j . Different from the former two settings, a regular distribution may have an unbounded support, which incurs extra technical challenges in proving the desired sample complexity of Algorithm 1.

To address this issue, we carefully truncate the given instance \mathcal{F} , such that (1) the resulting instance \mathcal{F}^* is still close to \mathcal{F} , under the measurement of the optimal Anonymous Reserve revenue; (2) \mathcal{F}^* has a small enough support supremum, which allows us to bound the revenue gap between it and its shaded counterpart $\widetilde{\mathcal{F}}^*$ (à la the proofs in the former two settings). Indeed, (3) $\widetilde{\mathcal{F}}^*$ is dominated by the shaded instance $\widetilde{\mathcal{F}}$ (derived directly from \mathcal{F}), thus $\mathsf{AR}(\widetilde{\mathcal{F}}^*) \leq \mathsf{AR}(\mathcal{F})$. Combining everything together completes the proof in this setting.

¹¹ Particularly, even if $r_{\mathcal{F}} \geq B$, we still have (Second Term) $= 0 \leq \sqrt{8\beta \cdot H} \cdot \int_{r_{\pi}}^{H} (1 - \mathcal{F}_2(x)) \cdot dx$.

Auxiliary Lemmas. To elaborate the truncation scheme, let us introduce several useful facts. Below, Lemma 13 might be known in the literature, yet we include a short proof for completeness. Notably, it only requires the distributions $\mathcal{F} = \{F_j\}_{j \in [n]}$ to be independent.

▶ **Lemma 13** (Order Statistics). For any product instance $\mathcal{F} = \{F_j\}_{j \in [n]}$, the highest CDF \mathcal{F}_1 and the second-highest CDF \mathcal{F}_2 satisfy that $1 - \mathcal{F}_2(v) \leq (1 - \mathcal{F}_1(v))^2$ for any value $v \in \mathbb{R}_{\geq 0}$.

Proof. After elementary algebra (see [43, Section 4]), one can easily check that the highest CDF $\mathcal{F}_1(v) = \prod_{j \in [n]} F_j(v)$ and the second-highest CDF

$$\begin{split} \mathcal{F}_2(v) &= \sum_{i \in [n]} \mathbf{Pr} \left[s_i \geq v \wedge (s_j < v, \forall j \neq i) \right] \\ &+ \mathbf{Pr} [s_j < v, \ \forall j \in [n]] \qquad \qquad (\mathrm{draw} \ \{s_j\}_{j=1}^n \ \mathrm{from} \ \{F_j\}_{i \in [n]}) \\ &= \mathcal{F}_1(v) \cdot \left[1 + \sum_{j \in [n]} \left(1/F_j(v) - 1 \right) \right] \\ &\geq \mathcal{F}_1(v) \cdot \left[1 + \sum_{j \in [n]} \ln \left(1/F_j(v) \right) \right] \qquad \qquad (\mathrm{as} \ z \geq \ln(1+z) \ \mathrm{when} \ z \in \mathbb{R}_{\geq 0}) \\ &= \mathcal{F}_1(v) \cdot \left(1 - \ln \mathcal{F}_1(v) \right) \qquad \qquad (\mathrm{as} \ \mathcal{F}_1(v) = \prod_{j \in [n]} F_j(v)) \\ &\geq \mathcal{F}_1(v) \cdot \left(2 - \mathcal{F}_1(v) \right). \qquad \qquad (\mathrm{as} \ \ln(1-z) \leq -z \ \mathrm{when} \ z \in [0,1]) \end{split}$$

We thus conclude the proof of Lemma 13 by rearranging the above inequality.

We safely scale the original instance $\mathcal{F} = \{F_j\}_{j \in [n]}$ so that $\max_{v \in \mathbb{R}_{\geq 0}} \{v \cdot (1 - \mathcal{F}_1(v))\} = 1$. Together with Lemma 13, this normalization leads to the following observations.

$$ho$$
 Claim 14. $\mathsf{AR}(\mathcal{F}) = \max_{r \in \mathbb{R}_{\geq 0}} \left\{ r \cdot \left(1 - \mathcal{F}_1(r) \right) + \int_r^{\infty} \left(1 - \mathcal{F}_2(x) \right) \cdot \mathrm{d}x \right\} \geq 1$.

ightharpoonup Claim 15. The highest CDF \mathcal{F}_1 is stochastically dominated by the equal-revenue CDF Φ_1 , namely $\mathcal{F}_1(v) \geq \Phi_1(v) \stackrel{\text{def}}{=} (1 - \frac{1}{v})_+$ for any value $v \in \mathbb{R}_{\geq 0}$.

ightharpoonup Claim 16. The second-highest CDF \mathcal{F}_2 is stochastically dominated by the CDF $\Phi_2(v) \stackrel{\text{def}}{=} (1 - \frac{1}{v^2})_+$, namely $\mathcal{F}_2(v) \ge \Phi_2(v)$ for any value $v \in \mathbb{R}_{\ge 0}$.

Truncation Scheme. Based on the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, we construct the truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$ as follows: for both $i \in \{1, 2\}$ and any value $v \in \mathbb{R}_{\geq 0}$,

$$\mathcal{F}_{i}^{*}(v) \stackrel{\text{def}}{=} \begin{cases} \mathcal{F}_{i}(v) & \text{when } \mathcal{F}_{i}(v) \leq 1 - (\varepsilon/4)^{i} \\ 1 & \text{when } \mathcal{F}_{i}(v) > 1 - (\varepsilon/4)^{i} \end{cases}$$
 (Truncation)

We immediately get two useful facts about the truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$.

 \triangleright Claim 17. For $i \in \{1, 2\}$, the truncated *i*-th highest CDF \mathcal{F}_i^* is dominated by the original *i*-th highest CDF \mathcal{F}_i . Thus, the shaded counterpart $\widetilde{\mathcal{F}}_i^* = \mathsf{S}_{\mathcal{F}}(\mathcal{F}_i^*)$ is dominated by $\widetilde{\mathcal{F}}_i = \mathsf{S}_{\mathcal{F}}(\mathcal{F}_i)$.

Proof. The first dominance $\mathcal{F}_i^* \preceq \mathcal{F}_i$ is obvious (by construction). The second dominance $\widetilde{\mathcal{F}}_i^* \preceq \widetilde{\mathcal{F}}_i$ also holds, because $S_{\mathcal{F}}(\cdot)$ is a non-decreasing function (see Proposition 3).

 \triangleright Claim 18. The truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$ has a support supremum of $s_u \leq 4/\varepsilon$.

Proof. As we certified in Lemma 13, for any value $v \in \mathbb{R}_{\geq 0}$, the highest and second-highest CDF's satisfy that $1 - \mathcal{F}_2(v) \leq (1 - \mathcal{F}_1(v))^2$. From this one can derive that

$$\mathcal{F}_2^{-1}(1-\varepsilon^2/16) < \mathcal{F}_1^{-1}(1-\varepsilon/4).$$

For each *i*-th highest CDF \mathcal{F}_i , we indeed truncate the particular $(\frac{\varepsilon}{4})^i$ -fraction of quantiles that correspond to the largest possible values. In view of the above inequality, the truncated second-highest CDF \mathcal{F}_2^* must have a smaller support supremum than the truncated highest CDF \mathcal{F}_1^* . Due to Claim 15, we further have $\Phi_1(s_u) \leq \mathcal{F}_1(s_u) = \mathcal{F}_1^*(s_u) \leq 1 - \varepsilon/4$. That is, $1 - 1/s_u \leq 1 - \varepsilon/4$ and thus $s_u \leq 4/\varepsilon$. This completes the proof of Claim 18.

Revenue Loss. Below, Lemma 19 shows that (Truncation) only incurs a small revenue loss.

▶ **Lemma 19** (Revenue Loss). The truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$ satisfies that

$$\mathsf{AR}(\mathcal{F}^*) \ge (1 - \frac{3}{4} \cdot \varepsilon) \cdot \mathsf{AR}(\mathcal{F}).$$

Proof. We adopt a hybrid argument. For brevity, let $\mathsf{AR}(r, \mathcal{F}_1 \uplus \mathcal{F}_2^*)$ be the resulting Anonymous Reserve revenue (under any reserve $r \in \mathbb{R}_{\geq 0}$) when only the second-highest CDF is truncated, and let \overline{r} be the optimal reserve for the *hybrid* instance $\mathcal{F}_1 \uplus \mathcal{F}_2^*$. The lemma comes from these two inequalities:

$$AR(\mathcal{F}^*) \ge (1 - \varepsilon/4) \cdot AR(\mathcal{F}_1 \uplus \mathcal{F}_2^*) \tag{2}$$

$$\geq (1 - \varepsilon/4) \cdot (1 - \varepsilon/2) \cdot \mathsf{AR}(\mathcal{F}). \tag{3}$$

In the remainder of the proof, we verify these two inequalities one by one.

Inequality (2). Under replacing the original highest CDF \mathcal{F}_1 with \mathcal{F}_1^* , we claim that

$$\exists (r \leq \overline{r}): \qquad r \cdot (1 - \mathcal{F}_1^*(r)) \geq (1 - \varepsilon/4) \cdot \overline{r} \cdot (1 - \mathcal{F}_1(\overline{r})). \tag{*}$$

The new reserve $r \in [0, \overline{r}]$ may not be optimal for the truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$. Based on the revenue formula and assuming inequality (\star) , we can infer inequality (2):

$$\begin{split} \mathsf{AR}(\mathcal{F}^*) &\geq \mathsf{AR}(r,\ \mathcal{F}^*) = r \cdot \left(1 - \mathcal{F}_1^*(r)\right) + \int_r^\infty \left(1 - \mathcal{F}_2^*(x)\right) \cdot \mathrm{d}x \\ &\geq r \cdot \left(1 - \mathcal{F}_1^*(r)\right) + \left(1 - \varepsilon/4\right) \cdot \int_{\overline{r}}^\infty \left(1 - \mathcal{F}_2^*(x)\right) \cdot \mathrm{d}x \qquad \text{(as } r \leq \overline{r}) \\ &\geq \left(1 - \varepsilon/4\right) \cdot \mathsf{AR}(\overline{r},\ \mathcal{F}_1 \uplus \mathcal{F}_2^*) \qquad \qquad \text{(inequality (\star))} \\ &= \left(1 - \varepsilon/4\right) \cdot \mathsf{AR}(\mathcal{F}_1 \uplus \mathcal{F}_2^*). \qquad (\overline{r} \text{ is optimal for $\mathcal{F}_1 \uplus \mathcal{F}_2^*$)} \end{split}$$

It remains to verify inequality (\star) . If $\mathcal{F}_1(\overline{r}) < 1 - \varepsilon/4$, by construction we have $\mathcal{F}_1^*(v) = \mathcal{F}_1(v)$ for any value $v \leq \overline{r}$. Clearly, inequality (\star) holds by employing the same reserve $r \leftarrow \overline{r}$.

From now on, we safely assume $\mathcal{F}_1(\overline{r}) = \prod_{j \in [n]} F_j(\overline{r}) \ge 1 - \varepsilon/4$. Inequality (\star) is enabled by the next Claim 20, which can be summarized from [1, Section 2].

ightharpoonup Claim 20. For any continuous regular distribution F_j and any value $\overline{r} \in \mathbb{R}_{\geq 0}$, define the parameter $a_j \stackrel{\text{def}}{=} \overline{r} \cdot (1/F_j(\overline{r}) - 1)$. Then, $F_j(v) \leq \frac{v}{v + a_j}$ for any value $v \in [0, \overline{r}]$, with the equality holds when $v = \overline{r}$.

Consider another auxiliary highest CDF $\mathcal{G}_1^*(v) \stackrel{\text{def}}{=} \prod_{j \in [n]} \frac{v}{v + a_j}$. In view of Claim 20, it suffices to show the following instead of inequality (\star) :

$$\exists (r \le \overline{r}): \qquad r \cdot (1 - \mathcal{G}_1^*(r)) \ge (1 - \varepsilon/4) \cdot \overline{r} \cdot (1 - \mathcal{F}_1(\overline{r})), \tag{\diamond}$$

We choose $r \leftarrow \mathcal{G}_1^{*-1}(1-\varepsilon/4)$. Since $\mathcal{G}_1^*(\overline{r}) = \mathcal{F}_1(\overline{r}) \geq 1-\varepsilon/4$ (by Claim 20 and our assumption) and \mathcal{G}_1^* is an increasing function, we do have $r \leq \overline{r}$. Let us bound the new reserve r from below:

$$1 - \varepsilon/4 = \mathcal{G}_1^*(r) = \prod_{j \in [n]} \frac{r}{r + a_j} \le \frac{r}{r + \sum_{j \in [n]} a_j} \qquad \Rightarrow \qquad r \ge (4/\varepsilon - 1) \cdot \sum_{j \in [n]} a_j.$$

Given this, we can accomplish inequality (\diamond) as follows:

$$\begin{split} \text{LHS of } (\diamond) &= r \cdot (\varepsilon/4) \geq (1 - \varepsilon/4) \cdot \sum_{j \in [n]} a_j \\ &= (1 - \varepsilon/4) \cdot \overline{r} \cdot \sum_{j \in [n]} \left(1/F_j(\overline{r}) - 1\right) & \text{(by definition of } a_j) \\ &\geq (1 - \varepsilon/4) \cdot \overline{r} \cdot \sum_{j \in [n]} \left(1 - F_j(\overline{r})\right) & \text{(as CDF } F_j(\overline{r}) \in [0, 1]) \\ &\geq (1 - \varepsilon/4) \cdot \overline{r} \cdot \left(1 - \prod_{j \in [n]} F_j(\overline{r})\right) = \text{RHS of } (\diamond), \end{split}$$

where the last inequality is because $\sum z_i \ge 1 - \prod (1 - z_i)$ when z_i 's are between [0, 1].

Inequality (3). Since the reserve $r_{\mathcal{F}}$ is optimal for the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ but may not for the hybrid instance $\mathcal{F}_1 \uplus \mathcal{F}_2^*$, we deduce from the revenue formula that

$$\begin{aligned} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\mathcal{F}_1 \uplus \mathcal{F}_2^*) &\leq \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}_1 \uplus \mathcal{F}_2) - \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}_1 \uplus \mathcal{F}_2^*) \\ &= \int_{r_{\mathcal{F}}}^{\infty} \left(\mathcal{F}_2^*(x) - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x. \end{aligned}$$

By construction, $0 \le \mathcal{F}_2^*(v) - \mathcal{F}_2(v) \le (\frac{\varepsilon}{4})^2$ for any value $v \in \mathbb{R}_{\ge 0}$. Also, it follows from Claim 16 that $\mathcal{F}_2(v) + \frac{1}{v^2} \ge 1 \ge \mathcal{F}_2^*(v)$. Apply both facts to the RHS of the above inequality:

$$\begin{split} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\mathcal{F}_1 \uplus \mathcal{F}_2^*) &\leq \int_0^\infty \left(\mathcal{F}_2^*(x) - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\leq \int_0^\infty \min\left\{ (\tfrac{\varepsilon}{4})^2, \ \tfrac{1}{x^2} \right\} \cdot \mathrm{d}x \\ &= \varepsilon/2 \leq (\varepsilon/2) \cdot \mathsf{AR}(\mathcal{F}), \end{split} \qquad \text{(Claim 14: } \mathsf{AR}(\mathcal{F}) \geq 1)$$

which gives inequality (3) after rearranging. This completes the proof of Lemma 19.

We now prove that, when the sample complexity $m \geq 11520\varepsilon^{-3} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 4)$, the optimal Anonymous Reserve revenue from the shaded truncated instance $\widetilde{\mathcal{F}}^* = \widetilde{\mathcal{F}}_1^* \uplus \widetilde{\mathcal{F}}_2^*$ is indeed close enough to that from the truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$.

▶ Lemma 21. The following holds for the shaded truncated instance $\widetilde{\mathcal{F}}^* = \widetilde{\mathcal{F}}_1^* \uplus \widetilde{\mathcal{F}}_2^*$:

$$AR(\widetilde{\mathcal{F}}^*) > AR(\mathcal{F}^*) - \varepsilon/4.$$

Proof. Denote by r^* the optimal reserve for the truncated instance $\mathcal{F}^* = \mathcal{F}_1^* \uplus \mathcal{F}_2^*$. Clearly, r^* is at most the support supremum of $s_u \leq 4/\varepsilon$ (see Claim 18), and may not be optimal for the shaded truncated instance $\widetilde{\mathcal{F}}^* = \widetilde{\mathcal{F}}_1^* \uplus \widetilde{\mathcal{F}}_2^*$. As illustrated in the former two settings, the function $S_{\mathcal{F}}(x) \leq x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta$ for any $x \in [0,1]$. Given these, ¹²

$$\begin{split} \mathsf{AR}(\mathcal{F}^*) - \mathsf{AR}(\widetilde{\mathcal{F}}^*) &\leq \mathsf{AR}(r^*, \ \mathcal{F}^*) - \mathsf{AR}(r^*, \ \widetilde{\mathcal{F}}^*) \quad (r^* \text{ may not be optimal to } \widetilde{\mathcal{F}}^*) \\ &= r^* \cdot \left(\mathsf{S}_{\mathcal{F}}(\mathcal{F}_1^*(r^*)) - \mathcal{F}_1^*(r^*) \right) + \int_{r^*}^{4/\varepsilon} \left(\mathsf{S}_{\mathcal{F}}(\mathcal{F}_2^*(x)) - \mathcal{F}_2^*(x) \right) \cdot \mathrm{d}x \\ &\leq (\mathrm{First \ Term}) + (\mathrm{Second \ Term}) + 28\beta \cdot \varepsilon^{-1}, \end{split} \tag{4}$$

where

(First Term)
$$\stackrel{\text{def}}{=} r^* \cdot \sqrt{8\beta \cdot \mathcal{F}_1(r^*) \cdot \left(1 - \mathcal{F}_1(r^*)\right)}$$
.
(Second Term) $\stackrel{\text{def}}{=} \int_{r^*}^{4/\varepsilon} \sqrt{8\beta \cdot \mathcal{F}_2^*(x) \cdot \left(1 - \mathcal{F}_2^*(x)\right)} \cdot dx$.

In the reminder of the proof, we quantify these two terms one by one.

¹²Note that the interval of integration can be safely truncated to the support supremum of $s_u \leq 4/\varepsilon$.

First Term. We infer from Claims 15 and 17 that the truncated highest CDF $\mathcal{F}_1^*(v) \geq 1 - \frac{1}{v}$ for any value $v \in \mathbb{R}_{>0}$. Additionally, of course $\mathcal{F}_1^*(v) \leq 1$. We thus have

$$\begin{split} \text{(First Term)} & \leq r^* \cdot \sqrt{8\beta \cdot 1 \cdot \left[1 - (1 - 1/r^*)\right]} \\ & = \sqrt{8\beta \cdot r^*} \leq \sqrt{32\beta \cdot \varepsilon^{-1}}. \end{split} \qquad \text{(as } r^* \leq s_u \leq 4/\varepsilon) \end{split}$$

Second Term. Based on Claims 16 and 17, for any value $v \in \mathbb{R}_{\geq 0}$, the truncated second-highest CDF $\mathcal{F}_2^*(v) \geq (1 - \frac{1}{v^2})_+$. Also, of course $\mathcal{F}_2^*(v) \leq 1$. For these reasons,

$$\begin{split} &(\text{Second Term}) \leq \int_0^{4/\varepsilon} \sqrt{8\beta \cdot \left(1 - \mathcal{F}_2^*(x)\right)} \cdot \mathrm{d}x & \text{(as } \mathcal{F}_2^*(x) \leq 1) \\ &\leq \int_0^1 \sqrt{8\beta} \cdot \mathrm{d}x + \int_1^{4/\varepsilon} \sqrt{8\beta \cdot \frac{1}{x^2}} \cdot \mathrm{d}x & \text{(as } \mathcal{F}_2^*(x) \leq (1 - \frac{1}{x^2})_+) \\ &= \sqrt{8\beta} + \sqrt{8\beta} \cdot \ln(4/\varepsilon) = \sqrt{8\beta} \cdot \ln(4e/\varepsilon) \end{split}$$

Plug the above two inequalities into inequality (4):

$$\begin{split} \mathsf{AR}(\mathcal{F}^*) - \mathsf{AR}(\widetilde{\mathcal{F}}^*) &\leq \sqrt{\frac{32\beta}{\varepsilon}} + \sqrt{8\beta} \cdot \ln(\frac{4e}{\varepsilon}) + \frac{28\beta}{\varepsilon} \\ &\leq \frac{\varepsilon}{\sqrt{90}} + \frac{\varepsilon^{3/2} \cdot \ln(4e/\varepsilon)}{\sqrt{360}} + \frac{7\varepsilon^2}{720} \qquad \text{(Part 3 of Claim 8: } \beta \leq \frac{\varepsilon^3}{2880}\text{)} \\ &\leq \frac{\varepsilon}{\sqrt{90}} + \frac{\ln(4e) \cdot \varepsilon}{\sqrt{360}} + \frac{7\varepsilon^2}{720} \qquad \qquad (\sqrt{\varepsilon} \cdot \ln(\frac{4e}{\varepsilon}) \leq \ln(4e) \text{ for } 0 < \varepsilon < 1\text{)} \\ &\leq \varepsilon/4 \qquad \qquad (\frac{1}{\sqrt{90}} + \frac{\ln(4e)}{\sqrt{360}} + \frac{7}{720} \approx 0.2409 < \frac{1}{4}\text{)} \end{split}$$

This completes the proof of Lemma 21.

The next Corollary 22 accomplishes the proof in the continuous regular setting.

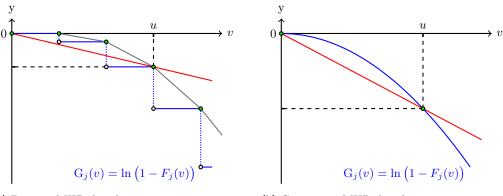
▶ Corollary 22. When the sample complexity $m \ge 11520\varepsilon^{-3} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 4)$:

$$\begin{split} \mathsf{AR}(\widetilde{\mathcal{F}}) &\geq \mathsf{AR}(\widetilde{\mathcal{F}}^*) \\ &\geq \mathsf{AR}(\mathcal{F}^*) - \varepsilon/4 \\ &\geq \mathsf{AR}(\mathcal{F}^*) - \varepsilon/4 \\ &\geq \mathsf{AR}(\mathcal{F}^*) - (\varepsilon/4) \cdot \mathsf{AR}(\mathcal{F}) \end{split} \qquad \begin{aligned} &(\mathsf{Claim} \ 17: & \mathsf{dominance} \ \widetilde{\mathcal{F}}_i \succeq \widetilde{\mathcal{F}}_i^*) \\ &\geq \mathsf{AR}(\mathcal{F}^*) - \varepsilon/4 \\ &\geq \mathsf{AR}(\mathcal{F}^*) - (\varepsilon/4) \cdot \mathsf{AR}(\mathcal{F}) \end{aligned} \qquad \begin{aligned} &(\mathsf{Claim} \ 14: & \mathsf{AR}(\mathcal{F}) \geq 1) \\ &\geq (1 - \varepsilon) \cdot \mathsf{AR}(\mathcal{F}). \end{aligned} \qquad (\mathsf{Lemma} \ 19: & \mathsf{AR}(\mathcal{F}^*) \geq (1 - \frac{3}{4} \cdot \varepsilon) \cdot \mathsf{AR}(\mathcal{F})) \end{aligned}$$

4.4 MHR Setting

In this subsection, we also assume that the original distributions $\mathcal{F} = \{F_j\}_{j \in [n]}$ are independent, and scale the instance such that $\max_{v \in \mathbb{R}_{\geq 0}} \{v \cdot (1 - \mathcal{F}_1(v))\} = 1$. Therefore, Lemma 13 and Claims 14–16 still holds. Nevertheless, the lower-bound formulas in Claims 15 and 16 (for the highest and second-highest CDF's) actually have too heavy tails. Namely, sharper formulas are required to prove the desired revenue gap between the original instance \mathcal{F} and its shaded counterpart $\widetilde{\mathcal{F}}$, given the more demanding sample complexity of $m = \mathcal{O}(\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$.

Based on the particular structures of the MHR distributions, we will first obtain workable lower-bound formulas, and then quantify the revenue loss between $\mathsf{AR}(\widetilde{\mathcal{F}})$ and $\mathsf{AR}(\mathcal{F})$. To this end, we safely assume $m \geq 5610\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 5)$.



- (a) Discrete MHR distribution
- (b) Continuous MHR distribution

Figure 2 Demonstration for the reduction in the proof of Lemma 23.

Lower-Bound CDF Formulas. Below, Lemma 23 shows that the highest and second-highest CDF's of any MHR instance decay exponentially fast.

- ightharpoonup Lemma 23. The following holds for any continuous or discrete MHR instance $\mathcal{F}=$ $\{F_i\}_{i\in[n]}$:
- 1. The highest CDF $\mathcal{F}_1(v) \geq 1 \frac{3}{2} \cdot e^{-v/6}$ for any value $v \geq e$. 2. The second-highest CDF $\mathcal{F}_2(v) \geq 1 \frac{9}{4} \cdot e^{-v/3}$ for any value $v \geq e$.
- **3.** The shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$ has a support supremum of $s_u \leq 12 \ln(\frac{21}{\varepsilon})$.

Proof. To see Item 1, we fix a parameter u > 1 (to be determined) and present a reduction (from the original MHR distributions $\mathcal{F} = \{F_i\}_{i \in [n]}$ to certain continuous exponential distributions) such that, for any value $v \geq u$, the highest CDF decreases point-wise.

We first handle the discrete MHR instances. As Figure 2a illustrates and by definition (see Section 2.1), such an instance $\mathcal{F} = \{F_j\}_{j \in [n]}$ has a discrete support of $\{k \cdot \Delta : k \in \mathbb{N}_{\geq 1}\}$, where the step-size $\Delta > 0$ is fixed. We must have $\Delta \leq 1$, because the instance is scaled so that $\max_{v \in \mathbb{R}_{>0}} \{v \cdot (1 - \mathcal{F}_1(v))\} = 1$ and Δ is exactly the support infimum (i.e., $\mathcal{F}_1(\Delta) = 0$).

For any $j \in [n]$, let us consider the step function $G_j(v) \stackrel{\text{def}}{=} \ln (1 - F_j(v))$ (marked in blue in Figure 2a) and the piece-wise linear function L_i (marked in gray) induced by the origin (0, 0) and the "¬"-type points $(k \cdot \Delta, G_j(k \cdot \Delta))$'s (marked in green). Apparently, $G_i(v) \leq L_i(v)$ for any value $v \in \mathbb{R}_{>0}$.

The MHR condition holds iff L_j is a concave function (see Section 2.1). Choose $u \leftarrow k \cdot \Delta$ (for some $k \in \mathbb{N}_{>1}$ to be determined) and let $a_j \stackrel{\text{def}}{=} -\frac{1}{u} \cdot G_j(u) > 0$, we infer from Figure 2a:

$$-a_j \cdot v \ge L_j(v) \ge G_j(v) = \ln\left(1 - F_j(v)\right) \qquad \Rightarrow \qquad F_j(v) \ge 1 - e^{-a_j \cdot v},\tag{5}$$

for any value $v \geq u$, with all the equalities holding when v = u. Given these, we also have

$$-a_j \cdot u = \ln\left(1 - F_j(u)\right) \le \ln\left(1 - \prod_{j \in [n]} F_j(u)\right) = \ln\left(1 - \mathcal{F}_1(u)\right),\tag{6}$$

for each $j \in [n]$. Put everything together: for any value $v \geq u$,

$$\begin{split} \ln \mathcal{F}_1(v) &= \sum_{j \in [n]} \ln F_j(v) \\ &\stackrel{(5)}{\geq} \sum_{j \in [n]} \ln (1 - e^{-a_j \cdot v}) \\ &= -\sum_{j \in [n]} \sum_{p=1}^{\infty} \frac{1}{p} \cdot e^{-p \cdot a_j \cdot u} \cdot e^{-p \cdot a_j \cdot u \cdot (v/u - 1)} \end{split} \tag{Taylor series)}$$

$$\stackrel{(6)}{\geq} -e^{\ln(1-\mathcal{F}_1(u))\cdot(v/u-1)} \cdot \sum_{j\in[n]} \sum_{p=1}^{\infty} \frac{1}{p} \cdot e^{-p\cdot a_j \cdot u} \quad (p\geq 1 \text{ and } v/u-1\geq 0)$$

$$= \left(1-\mathcal{F}_1(u)\right)^{v/u-1} \cdot \ln \prod_{j\in[n]} (1-e^{-a_j \cdot u}) \qquad \text{(Taylor series)}$$

$$\stackrel{(5)}{=} \left(1-\mathcal{F}_1(u)\right)^{v/u-1} \cdot \ln \mathcal{F}_1(u), \qquad \text{(equality condition)}$$

from which we deduce that $\mathcal{F}_1(v) \geq (\mathcal{F}_1(u))^{(1-\mathcal{F}_1(u))^{v/u-1}}$ for any value $v \geq u = k \cdot \Delta$. It can be seen that this lower-bound formula is an increasing function in the term $\mathcal{F}_1(u) \in [0,1]$.

We would like to choose $k \leftarrow \lfloor e/\Delta \rfloor$. Because the step-size $\Delta \leq 1$, we do have $k \in \mathbb{N}_{\geq 1}$ and $u = k \cdot \Delta \in [e-1, e]$. Then, it follows from Claim 15 that $\mathcal{F}_1(u) \geq \mathcal{F}_1(e-1) \geq 1 - \frac{1}{e-1}$. Replace the term $\mathcal{F}_1(u)$ in the above lower-bound formula with this bound:

$$\begin{split} \mathcal{F}_1(v) & \geq \left(\frac{e-2}{e-1}\right)^{(e-1)^{1-v/u}} = e^{-\ln(\frac{e-1}{e-2})\cdot(e-1)^{1-v/u}} \\ & \geq 1 - \ln(\frac{e-1}{e-2})\cdot(e-1)^{1-v/u} & \text{(as } e^{-z} \geq 1-z) \\ & \geq 1 - \ln(\frac{e-1}{e-2})\cdot(e-1)^{1-v/e} & \text{(as } u \leq e) \\ & = 1 - (e-1)\cdot\ln(\frac{e-1}{e-2})\cdot e^{-\frac{\ln(e-1)}{e}\cdot v} \\ & \geq 1 - \frac{3}{2}\cdot e^{-v/6}, & \text{(elementary algebra)} \end{split}$$

for any value $v \in [u, \infty)$. Clearly, this inequality holds in the shorter range of $v \in [e, \infty)$.

When $\mathcal{F} = \{F_j\}_{j \in [n]}$ is a continuous MHR instance, by definition (see Section 2.1) each function $G_j(v) = \ln (1 - F_j(v))$ itself is a concave function (as Figure 2b illustrates). That is, we can simply choose $u \leftarrow e$ and apply the same arguments as the above. Actually, we can get a better lower-bound formula that $\mathcal{F}_1(v) \geq 1 - \frac{5}{4} \cdot e^{-v/e}$ for any value $v \geq e$.

Clearly, **Item 2** is an implication of **Item 1** and Lemma 13. Now, we turn to attesting **Item 3**. By definition, the function $S_{\mathcal{F}}(x) = \min \{1, x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta\} = 1$ when $x \geq 1 - 7\beta$. Hence, the shaded instance $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$ has a support supremum of

$$\begin{split} s_u & \leq \max\{\mathcal{F}_1^{-1}(1-7\beta), \quad \mathcal{F}_2^{-1}(1-7\beta)\} \\ & = \mathcal{F}_1^{-1}(1-7\beta) \leq 6\ln(\frac{3}{14} \cdot \beta^{-1}) \\ & \leq 6\ln(\frac{2805}{7\varepsilon^2}) \leq 12\ln(21/\varepsilon). \end{split} \qquad \text{(Part 1 of Lemma 23)}$$

This completes the proof of Lemma 23.

Revenue Loss. Conceivably, the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ should have a small optimal reserve $r_{\mathcal{F}}$, since \mathcal{F}_1 and \mathcal{F}_2 both have *light tails*. This proposition is formalized as the next Lemma 24, which will be useful in our later proof.

▶ Lemma 24. For the original MHR instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, there is an optimal Anonymous Reserve auction having a reserve of $r_{\mathcal{F}} \leq \mathcal{C}^*$, where the constant $\mathcal{C}^* \approx 20.5782$ is the larger one between the two roots of the transcendental equation $\frac{3}{2} \cdot z \cdot e^{-z/6} = 1$.

Proof. The proof here is similar in spirit to that of Claim 10. When there are multiple alternative optimal reserves $r_{\mathcal{F}}$'s, we would select the *smallest* one. To see the lemma, we need the math fact $\frac{3}{2} \cdot z \cdot e^{-z/6} < 1$ when $z > \mathcal{C}^* \approx 20.5782$. Then, it follows from Part 1 of Lemma 23 that

$$r \cdot \left(1 - \mathcal{F}_1(r)\right) \le \frac{3}{2} \cdot r \cdot e^{-r/6} < 1,\tag{7}$$

for any reserve $r > \mathcal{C}^*$. Particularly, $\lim_{r \to \infty} r \cdot (1 - \mathcal{F}_1(r)) = 0$. By contrast, we have scaled the instance such that $\max_{v \in \mathbb{R}_{\geq 0}} \left\{ v \cdot \left(1 - \mathcal{F}_1(v)\right) \right\} = 1$, which means $\overline{r} \cdot \left(1 - \mathcal{F}_1(\overline{r})\right) = 1$ for some other reserve $\overline{r} \in [0, \mathcal{C}^*]$. Recall the Anonymous Reserve revenue formula:

$$\begin{split} \mathsf{AR}(\overline{r},\ \mathcal{F}) - \mathsf{AR}(r,\ \mathcal{F}) &= 1 - r \cdot \left(1 - \mathcal{F}_1(r)\right) + \int_{\overline{r}}^r \left(1 - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\geq 1 - r \cdot \left(1 - \mathcal{F}_1(r)\right) \overset{(7)}{\geq} 0. \end{split} \qquad \text{(as } \overline{r} \leq \mathcal{C}^* < r) \end{split}$$

That is, under our tie-breaking rule, any reserve $r > \mathcal{C}^*$ cannot be revenue-optimal. Apparently, this observation indicates Lemma 24.

Finally, Lemma 25 establishes the desired revenue gap between the original instance $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$ and its shaded counterpart $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \uplus \widetilde{\mathcal{F}}_2$, thus settling the MHR case.

▶ **Lemma 25.** When the sample complexity $m \ge 5610\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1} + 5)$:

$$\mathsf{AR}(\widetilde{\mathcal{F}}) \ge (1 - \varepsilon) \cdot \mathsf{AR}(\mathcal{F}).$$

Proof. Recall that the function $S_{\mathcal{F}}(x) \leq x + \sqrt{8\beta \cdot x \cdot (1-x)} + 7\beta$ when $x \in [0,1]$. Based on the support supremum $s_u \leq 12 \ln(\frac{21}{\varepsilon})$ established in Part 3 of Lemma 23 and the Anonymous Reserve revenue formula, we deduce that 13

$$\begin{split} \mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\widetilde{\mathcal{F}}) &\leq \mathsf{AR}(r_{\mathcal{F}}, \ \mathcal{F}) - \mathsf{AR}(r_{\mathcal{F}}, \ \widetilde{\mathcal{F}}) \qquad (r_{\mathcal{F}} \ \mathsf{may} \ \mathsf{not} \ \mathsf{be} \ \mathsf{optimal} \ \mathsf{to} \ \widetilde{\mathcal{F}}) \\ &= r_{\mathcal{F}} \cdot \left(\widetilde{\mathcal{F}}_1(r_{\mathcal{F}}) - \mathcal{F}_1(r_{\mathcal{F}})\right) + \int_{r_{\mathcal{F}}}^{12 \ln(21/\varepsilon)} \left(\widetilde{\mathcal{F}}_2(x) - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\quad + \int_{12 \ln(21/\varepsilon)}^{\infty} \left(1 - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \\ &\leq \left(\mathsf{First} \ \mathsf{Term}\right) + \left(\mathsf{Second} \ \mathsf{Term}\right) + \left(\mathsf{Third} \ \mathsf{Term}\right) + 84\beta \cdot \ln(21/\varepsilon), \quad (8) \end{split}$$

where

(First Term)
$$\stackrel{\text{def}}{=} r_{\mathcal{F}} \cdot \sqrt{8\beta \cdot \mathcal{F}_{1}(r_{\mathcal{F}}) \cdot \left(1 - \mathcal{F}_{1}(r_{\mathcal{F}})\right)}$$
.
(Second Term) $\stackrel{\text{def}}{=} \int_{r_{\mathcal{F}}}^{12 \ln(21/\varepsilon)} \sqrt{8\beta \cdot \mathcal{F}_{2}(x) \cdot \left(1 - \mathcal{F}_{2}(x)\right)} \cdot dx$.
(Third Term) $\stackrel{\text{def}}{=} \int_{12 \ln(21/\varepsilon)}^{\infty} \left(1 - \mathcal{F}_{2}(x)\right) \cdot dx$.

In the reminder of the proof, we quantify these three terms one by one.

[First Term]. Recall Claim 15 that the highest CDF $\mathcal{F}_1(v) \geq 1 - \frac{1}{v}$ for any value $v \in \mathbb{R}_{\geq 0}$. Further, of course $\mathcal{F}_1(v) \leq 1$. Given these and because $r_{\mathcal{F}} \leq \mathcal{C}^* \approx 20.5782$ (see Lemma 24),

(First Term)
$$\leq r_{\mathcal{F}} \cdot \sqrt{8\beta \cdot 1 \cdot \left[1 - (1 - 1/r_{\mathcal{F}})\right]} = \sqrt{8\beta \cdot r_{\mathcal{F}}} \leq \sqrt{165\beta}$$
.

[Second Term]. Clearly, $\mathcal{F}_2(v) \in [0,1]$ for all value $v \in \mathbb{R}_{\geq 0}$. Additionally, we infer from Part 2 of Lemma 23 that $\sqrt{1 - \mathcal{F}_2(v)} \leq \frac{3}{2} \cdot e^{-v/6}$ when $v \geq e$. For these reasons,

(Second Term)
$$\leq \int_0^\infty \sqrt{8\beta \cdot \mathcal{F}_2(x) \cdot \left(1 - \mathcal{F}_2(x)\right)} \cdot dx$$

 $\leq \int_0^e \sqrt{8\beta} \cdot dx + \int_e^\infty \sqrt{8\beta} \cdot \frac{3}{2} \cdot e^{-x/6} \cdot dx$
 $= \sqrt{8\beta} \cdot (e + 9e^{-e/6}) \leq \sqrt{570\beta}$ (elementary algebra)

¹³ Note that the interval of integration can be safely truncated to the support supremum of $s_u \leq 12 \ln(\frac{21}{\epsilon})$.

[Third Term]. Also, we deduce from Part 2 of Lemma 23 that

$$(\text{Third Term}) = \int_{12\ln(21/\varepsilon)}^{\infty} \left(1 - \mathcal{F}_2(x)\right) \cdot \mathrm{d}x \leq \int_{12\ln(21/\varepsilon)}^{\infty} \frac{9}{4} \cdot e^{-x/3} \cdot \mathrm{d}x = \frac{\varepsilon^4}{28812} \leq \frac{\varepsilon}{28812}.$$

Plug the above three inequalities into inequality (8):

$$\begin{split} &\mathsf{AR}(\mathcal{F}) - \mathsf{AR}(\widetilde{\mathcal{F}}) \\ &\leq \sqrt{165\beta} + \sqrt{570\beta} + \frac{\varepsilon}{28812} + 84\beta \cdot \ln(21/\varepsilon) \\ &\leq \sqrt{\frac{3}{34}}\varepsilon + \sqrt{\frac{57}{187}}\varepsilon + \frac{\varepsilon}{28812} + \frac{42\varepsilon^2 \cdot \ln(21/\varepsilon)}{935} & \text{(Claim 8, Part 4: } \beta \leq \frac{\varepsilon^2}{1870}) \\ &\leq \sqrt{\frac{3}{34}}\varepsilon + \sqrt{\frac{57}{187}}\varepsilon + \frac{\varepsilon}{28812} + \frac{42\ln 21}{935}\varepsilon & (\varepsilon \cdot \ln(\frac{21}{\varepsilon}) \leq \ln 21 \text{ for } 0 < \varepsilon < 1) \\ &\leq \varepsilon \leq \varepsilon \cdot \mathsf{AR}(\mathcal{F}) & \text{(elementary algebra)} \end{split}$$

where the last inequality is due to Claim 14. This completes the proof of Lemma 25.

4.5 Continuous λ -Regular Setting

In the literature, there is another distribution family that receives much attention [16, 24, 23, 2] – the continuous λ -regular distributions. When the built-in parameter λ ranges from 0 to 1, this family smoothly expands from the MHR family to the regular family.

À la the MHR case, the sample complexity upper bound is still $\mathcal{O}(\varepsilon^{-2} \cdot (\ln \varepsilon^{-1} + \ln \delta^{-1}))$, despite that the $\mathcal{O}(\cdot)$ notation now hides some absolute constant \mathcal{C}_{λ} depending on $\lambda \in (0,1)$. Since the proof of this bound is very similar to the MHR case, we just show in the full version a counterpart extreme value theorem (cf. Lemma 23), but omit the other parts about the revenue smoothness analysis.

It is noteworthy that the $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ upper bound may not be optimal. Namely, in both of the continuous λ -regular setting and the continuous MHR setting, the best known lower bounds are $\Omega(\varepsilon^{-3/2})$ [39]. It would be interesting to pin down the exact sample complexity in both settings, for which the tools developed here and by [14, 39, 35] might be useful.

5 Conclusion and Further Discussion

In this work, we proved the nearly tight sample complexity of the Anonymous Reserve auction, for each of the $[0,1]^n$ -bounded, $[1,H]^n$ -bounded, regular and MHR distribution families. In the literature on "mechanism design via sampling", a notion complementary to sample complexity is regret minimization (e.g., see [8, 12, 11] and the follow-up papers). Regarding the Anonymous Reserve auction, this means the seller must select a careful reserve price $r_t \in \mathbb{R}_{\geq 0}$ in each round t over a time horizon $T \in \mathbb{N}_{\geq 1}$, in order to maximize the cumulative revenue, i.e., minimize the cumulative revenue loss against a certain benchmark.

Indeed, if the seller can access the highest and second-highest bids in all of the past (t-1) rounds, our results imply the nearly optimal regret bounds. Consider the [0,1]-additive setting for example. Because $\mathcal{O}(\varepsilon^{-2} \cdot \ln \varepsilon^{-1})$ samples suffice to reduce the revenue loss to $\varepsilon \in (0,1)$, the regret in each round $t \in [T]$ is at most $\mathcal{O}(\sqrt{(\ln t)/t})$. As a result, the cumulative regret is at most $\sum_{t=1}^T \mathcal{O}(\sqrt{(\ln t)/t}) = \mathcal{O}(\sqrt{T} \cdot \ln T)$. Similarly, the $\Omega(\varepsilon^{-2})$ lower bound on the sample complexity implies an $\Omega(\sqrt{T})$ lower bound on the regret.

[17] considered the same problem under weaker data access, where the seller can only observe the allocations and the payments in the past (t-1) rounds. This models some particular markets, where the seller is not the auctioneer and can acquire a least amount of information. Assuming the bids are i.i.d. and supported on [0,1], Cesa-Bianchi et al. proved

a matching regret of $\widetilde{\Theta}(\sqrt{T})$.¹⁴ But, what if the seller still has the weak data access yet the distributions are distinct and even correlated? The regret of the Anonymous Reserve auction and other mechanisms in this model is an interesting problem.

Additionally, another natural and meaningful adjusted model is to assume that the bidders would *strategically* report their samples, or further, that the bidders themselves are *learners* as well. At the time of our paper, this research direction is very nascent yet has already received much attention. For an overview of this, the reader can turn to [13, 44, 38, 31] and the references therein.

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¹⁴More precisely, their upper bound is $\mathcal{O}(\sqrt{T \cdot \ln \ln \ln T} \cdot \ln \ln T)$ and their lower bound is also $\Omega(\sqrt{T})$.

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