The Value of Information Concealment

Hu Fu*   Christopher Liaw*   Pinyan Lu†   Zhihao Gavin Tang‡

Abstract

We consider a revenue optimizing seller selling a single item to a buyer, on whose private value the seller has a noisy signal. We show that, when the signal is kept private, arbitrarily more revenue could potentially be extracted than if the signal is leaked or revealed. We then show that, if the seller is not allowed to make payments to the buyer and if the value distribution conditioning on each signal is regular, the gap between the two is bounded by a multiplicative factor of 3. We give examples showing that both conditions are necessary for a constant bound on the gap to hold.

We connect this scenario to multi-bidder single-item auctions where bidders' values are correlated. Similarly to the setting above, we show that the revenue of a Bayesian incentive compatible, ex post individually rational auction can be arbitrarily larger than that of a dominant strategy incentive compatible auction, whereas the two are no more than a factor of 5 apart if the auctioneer never pays the bidders and if the distribution is jointly regular. The upper bounds in both settings degrade gracefully when the distribution is a mixture of a small number of regular distributions.

1 Introduction

Revenue maximization in the sale of a single item to a single buyer is a fundamental problem in mechanism design. In the classical model, the buyer holds a private value v for the item, and the seller only knows the distribution $\mathcal{D}$ from which v is drawn. The revenue-maximizing strategy of the seller is to offer the buyer an optimal take-it-or-leave-it price. This simple fact is one of the earliest insights offered by the revenue optimization literature [15, 17], and also one of the key building blocks of many recent approximation results in more complicated scenarios [e.g. 7, 2].

Implicit in this classical model is a simple informa-

tion structure: the buyer is completely informed, and the seller only partially so. As a consequence, when taking a price, the buyer usually derives a positive utility knows as the *information rent* due to this information asymmetry. In modern online markets, however, there is often information asymmetry in the other direction as well. Even if the buyer always knows fully her value (which may or may not be the case in practice), the seller may still possess partial information of which the buyer cannot be sure. For example, transactions in the near past with other buyers may suggest market trends which inform the seller but are not precisely known to the buyer. The notorious obscurity of machine learning techonologies also contributes to such asymmetry — whereas the seller may have more or less accurate estimates on the buyer's value based on various observable attributes, the buyer may not be able to determine which attributes of hers are picked up as cues, and hence could not determine what estimate the seller has. In these scenarios, the buyer is uncertain about the information held by the seller, even though she is completely informed of her value, seemingly the central information in the transaction.

This paper studies the implication, on the seller's revenue, of information asymmetry of this kind. The basic question we raise is: *Are there nontrivial ways for the seller to exploit this asymmetry so as to increase his revenue?* The answer, in short, is that sophisticated exploitation is possible and in some cases could drive up the revenue unboundedly compared with "naïve" uses of this information asymmetry; however, under a few conditions that seem naturally satisfied in practical scenarios, the benefit of such sophistication is bounded by a small muliplicative constant factor. Below we explain what we mean by "sophisticated" and "naïve" pricings, after we lay down the basic model.

The Model. We model the simplest information asymmetry by having a value v and a signal s drawn from a commonly known correlated distribution; the buyer only sees her value v for the item and the seller only sees s. The seller aims to optimize his revenue, taking expectation over the joint distribution.

Naïve pricing: Upon observing s, the seller has a distribution over v conditioning on s. We say the seller naïvely uses the signal if he simply adopts the optimal

---
*Department of Computer Science, University of British Columbia. {hufu, cvliaw}@cs.ubc.ca

†School of Information Management and Engineering, Shanghai University of Finance and Economics. lu.pinyan@mail.shufe.edu.cn

‡Department of Computer Science, University of Hong Kong. zhtang@cs.hku.hk

strategy for each such conditional distribution, which amounts to posting a take-it-or-leave-it price optimally for each s. This is also the best strategy of the seller if s is publicized or leaked. In an e-commerce scenario, this could be the case where the seller publicizes his learning algorithm or whichever information gathered pertaining to the buyer.

It may come as a surprise that the naïve pricing may not be optimal; after all, in a sense, it optimizes the revenue pointwise. The suboptimality of this approach lies precisely in its failure to exploit information asymmetry. Publicizing the signal reduces its worth to a mere sharper estimte of the value distribution, and gives up its uncertainty, when kept private, in the buyer's eyes. We show in Section 3.1 that *this uncertainty can be leveraged to elicit more information from the buyer and hence increase the seller's revenue, in some cases by an arbitrarily large multiplicative factor.*

Main result. Our upper bound on the worth of concealing the signal is the most technically-sophisticated part of the paper. Under two conditions, we show that the optimal revenue with private signal is no more than 3 times the revenue of optimal posted price under each (public) signal. The first condition asks that the seller never pay money to the buyer, which seems a natural requirement in many scenarios. The second condition requires that each conditional distribution of the value v is *regular* in Myerson [15]'s sense.[1] Viewed alternatively, our bound suggests that a seller looking to boost the value of private signals may consider making payments to the buyer occassionally or sharpening the signals in such a way that the resulting conditional value distributions have interleaved supports (see Example 2 for an instance of this).

We remark that, in the literature on revenue optimzation, regularity of value distributions usually does not mitigate the gaps between revenues of different mechanisms, not least because the most common distribution used in showing such large gaps, the equal revenue distribution, is itself regular. As we explain later, the regularity condition enters our analysis in a non-standard way.

Regularity is often deemed a natural condition when the buyer can be identified as a certain category of customer, whereas uncertainty on the category results in irregular distributions that is a mixture of regular distributions [see, e.g., 19]. Our upper bound degrades gracefully for mixtures of a small number of regular distributions.

Connection to multi-bidder auctions. We are not the first to observe that correlated external information can be used to extract the buyer's information rent. For multi-bidder auctions where the bidders' values are drawn from correlated distributions, classical works by Crémer and McLean [8, 9] showed ways to extract from each bidder her full value by leveraging her competitors' bids. The similarity between the two settings is immediate when one sees the competitors' bids as private signals. However, Crémer and McLean's results do not apply to us because we constrain our seller's mechanism much more stringently: crucial for Crémer and McLean's mechanisms is that the auctioneer may charge the buyer anything as long as the buyer, conditioning on any value she holds, has nonnegative utility *in expectation* (over the signals or her competitors' bids). This is known as *interim* individual rationality (IR). In contrast, we insist on the more realistic *ex post individual rationality*, which stipulates that the buyer should never have a negative utility no matter what signal is realized. This key difference is what makes our examples of unbounded gaps more nontrivial and our constant upper bound possible.

In multi-bidder auctions, the solution concept corresponding to naïve pricing or pricing with publicized signals is *dominant strategy incentive compatibility* (DSIC), whereas the more sophisticated selling with private signals corresponds to *Bayesian incentive compatible* (BIC) auctions. The second main contribution of this paper is the first study of the revenues of DSIC and BIC single-item auctions, which complements several recent works that compare the revenues of DSIC and BIC *multi-item* auctions [20, 22]. (Under the looser interim IR constraint, a Crémer and McLean [9] shows a gap between the optimal DSIC and BIC revenues, and recently Albert et al. [1] fully characterized the gap.)

We show an unbounded gap between the revenues of DSIC and BIC auctions (under ex post individual rationality); with non-negative payments and regular distributions, we show that Ronen [18]'s DSIC lookahead auction 5-approximates the optimal BIC revenue. The first part follows immediately from the single-buyer examples, whereas the 5-approximation is technically more nontrivial, as we discuss below.

Our techniques. The driving horse for our main result is the duality framework which was recently promoted by Cai et al. [4] and subsequently applied to various settings [e.g. 3, 12, 5, 14, 10]. The basic approach is to write the optimal revenue of a target mechanism (in our case, the revenue with private signals, or the optimal BIC revenue) as the objective of a linear program and then Lagrangify the IC and IR constraints; the value of the resulting Lagrangian instantiated with

---

[1]Regularity is a standard notion in the literature of revenue optimization, to be precisely defined in Section 2.

any dual variables serves as an upper bound on the optimal revenue and can be used as a benchmark for approximation. What is peculiar about the argument in this work is that, since the the signals' uncertainty is key to revenue maximization, the dual variables controlling the buyer's incentives cannot be engineered for each value distribution conditioned on a signal. That is, these dual variables cannot be functions of the signals. On the other hand, standard dual variables giving rise to the public signal revenue via Myerson's *virtual values* are distribution dependent. In general, distribution-independent dual variables do not give rise to upper bounds that can be bounded by the virtual values. We overcome this difficulty by constructing dual variables which, besides the "virtual value terms", generate another term that grows in the worst case with the welfare. A fairly non-standard argument shows that regularity brings this extra term down to comparable with naïve pricing revenues (or DSIC revenue in multi-bidder auctions).

Multi-bidder auctions pose another difficulty. The seller not only maximizes the revenue from each bidder, but is also subject to the item's availability; two bidders cannot both be assigned the item. Revenue optimal auctions are already complicated even under the more stringent DSIC constraint and is unlikely to have a succinct characterzation [e.g. 11, 16, 6]. One way to disentangle multiple bidders' interaction is provided by Ronen [18]'s DSIC *lookahead auction*, which maximizes the revenue from the highest bidders and gives up lower bidders. Ronen showed that this 2-approximates the optimal DSIC revenue. However, the natural extension of Ronen's auction to the BIC auctions fails, due to a subtle reason we explain briefly at the end of the paper (Section 5). Instead, we take the duality approach again, and appeal to a property possessed by the particular dual variables we constructed for the single buyer case (Claim 4.1 in Section 4). This allows us to bound the lower bidders' contribution in the Langrangian by twice the second price auction revenue, and the problem boils down to maximizing revenue from the highest bidder, which is a single buyer problem, where we could apply our previous argument. We note that bounding lower bidders' contribution to revenue by the second highest value is the essence of Ronen [18]'s original proof, and the observation that this can be done in the Langragian was implicit in several duality-based works [e.g. 4, 13].

2 Preliminaries

Single-Buyer Pricing with Signals. A seller tries to sell a single item to a buyer, who has a private type t, which indicates the buyer's value of the item; throughout the paper we use type t and value v interchangeably. The seller observes a private signal $s \in S$. The pair (t, s) is drawn from a commonly known joint distribution $\mathcal{D}$, whose density function we denote by $f(\cdot, \cdot)$. For a fixed signal s, we use $\mathcal{D}(t|s)$ to denote the conditional distribution of t given the signal s and denote its density function by $f(t|s)$. We also write $F(t|s) := \int_{t' \leq t} f(t'|s)\, dt'$ as the conditional cumulative density function. A selling mechanism consists of an allocation rule $x(v, s)$, indicating the probability with which the buyer gets the item when she reports her value as v and the seller sees signal s, and a payment rule $p(v, s)$, the payment made by the buyer to the seller. We say the mechanism has *no negative payment* if p is never negative, i.e., the seller never pays the buyer. The expected revenue of a mechanism is $\mathbf{E}_{(v,s)\sim\mathcal{D}}[p(v, s)]$.

In *naïve pricing*, or *pricing with publicized or leaked signals*, the seller adopts, for each observed s, a mechanism for the value distribution conditioned on s. Such a mechanism needs to be (i) ex post individually rational (IR): for any v and s, $vx(v, s) - p(v, s) \geq 0$, and (ii) incentive compatible (IC): for any v, v' and s, $v \cdot x(v, s) - p(v, s) \geq v \cdot x(v', s) - p(v', s)$.

DEFINITION 2.1. (VIRTUAL VALUES) *Given a value distribution with density function f and cumulative density function F, the* Myerson virtual value *is $\varphi(v) := v - \frac{1-F(v)}{f(v)}$. Given a joint distribution $\mathcal{D}$ on (v, s), we denote by $\varphi(v|s)$ the virtual value of v in the conditional distribution $F(v|s)$.*

LEMMA 2.1. (MYERSON 1981) *When the value distribution is F (with no signal), a mechanism is IC only if its allocation rule x is monotone non-decreasing in v. Its expected revenue is the* virtual surplus *$\int_v \varphi(v)x(v)\, dF(v)$. The revenue-optimal mechanism for any F posts a take-it-or-leave-it price.*

Lemma 2.1 implies that with public signals, posting a price which is optimal for each signal s maximizes the seller's revenue.

For mechanisms with *private signals*, besides being (i) ex post IR, the IC constraint weakens to: (ii') for all v and v', $\mathbf{E}_{s\sim\mathcal{D}(s|v)}[v \cdot x(v, s) - p(v, s)] \geq \mathbf{E}_{s\sim\mathcal{D}(s|v)}[v \cdot x(v', s) - p(v', s)]$, where $\mathcal{D}(s|v)$ denotes the conditional distribution of s given v. Equivalently, the constraint can be written as $\int_s (v \cdot x(v, s) - p(v, s))f(v, s)\, ds \geq \int_s (v \cdot x(v', s) - p(v', s))f(v, s)\, ds$.

DEFINITION 2.2. *A value distribution F (without signals) is regular if its support is an interval $[l, h] \subset \mathbb{R}$, and its virtual value φ is nondecreasing on $[l, h]$. A distribution $\mathcal{D}$ on (v, s) is jointly regular if the conditional distribution of v given any signal s is regular.*

Single-item (Multi-bidder) Auctions. In a single item auction, one indivisible good is to be allocated to at most one of n bidders. Bidder i has a private type $t_i \in \mathbb{R}$, and the types are drawn from a possibly correlated, commonly known joint distribution $\mathcal{D}$. We write $\boldsymbol{t} = (t_1, \ldots, t_n)$ to denote the profile of types and $\boldsymbol{t}_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. A mechanism (auction) is specified by its allocation rules $x_i \colon \boldsymbol{t} \mapsto [0,1]$ and payment rules $p_i \colon \boldsymbol{t} \mapsto \mathbb{R}$, for each bidder i. The utility of bidder i is $u_i(\boldsymbol{t}) = t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t})$. The allocation rules are required to satisfy the feasbility constraint: $\sum_i x_i(\boldsymbol{t}) \leq 1$, for all $\boldsymbol{t}$ and i.

Similar to the IC constraint under public signals in single buyer pricing, a mechanism is *dominant strategy incentive compatible* (DSIC) if for each bidder i, each type profile $\boldsymbol{t}$, and any deviation t_i',

$$t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t}) \geq t_i \cdot x_i(t_i', \boldsymbol{t}_{-i}) - p_i(t_i', \boldsymbol{t}_{-i}).$$

Similar to the IC constraint under private signals, an auction is *Bayesian incentive compatible* (BIC) if for each bidder i with type t_i and possible deviation t_i',

$$\mathbf{E}_{\boldsymbol{t}_{-i} \sim \mathcal{D}(\boldsymbol{t}_{-i}|t_i)} \left[t_i \cdot x_i(\boldsymbol{t}) - p(\boldsymbol{t}) \right]$$
$$\geq \mathbf{E}_{\boldsymbol{t}_{-i} \sim \mathcal{D}(\boldsymbol{t}_{-i}|t_i)} \left[t_i \cdot x_i(t_i', \boldsymbol{t}_{-i}) - p(t_i', \boldsymbol{t}_{-i}) \right],$$

where $\mathcal{D}(\boldsymbol{t}_{-i}|t_i)$ denotes the distribution of $\boldsymbol{t}_{-i}$ conditioned on bidder i's type being t_i. A mechanism is *ex post individually rational* (IR) if, for any bidder i and any type profile $\boldsymbol{t}$, $t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t}) \geq 0$ and *interim IR* if $\mathbf{E}_{\boldsymbol{t}_{-i} \sim \mathcal{D}(\boldsymbol{t}_{-i}|t_i)}[t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t})] \geq 0$ for any bidder i and her type t_i.

Myerson [15]'s seminal paper showed that, for a product distribution $\mathcal{D}$, a DSIC, ex post IR auction maximizes revenue among all BIC, interim IR auction. In sharp contrast, for correlated distributions, Crémer and McLean [8, 9] showed that, except for degenerate cases, interim IR auctions can extract full surplus, i.e., $\mathbf{E}_{\boldsymbol{t}}[\max_i t_i]$. Throughout this paper we require the mechanisms to be ex post IR.

The Lookahead Auction. Ronen [18]'s lookahead auction reduces multi-bidder DSIC revenue maximization to pricing for a single bidder, while retaining at least half of the optimal revenue.

DEFINITION 2.3. ((DOMINANT STRATEGY) LOOKAHEAD AUCTION) *For each bid vector $\boldsymbol{t}$, the lookahead auction chooses the highest bidder i^*, and offers an optimal take-it-or-leave-it price conditioning on (i) $\boldsymbol{t}_{-i^*}$ and (ii) t_{i^*} being the highest type.*

THEOREM 2.1. (RONEN 2001) *The dominant strategy lookahead auction is DSIC and ex post IR, and extracts at least half of the revenue of the optimal DSIC, ex post IR auction.*

We designate Ronen's auction as the *dominant strategy* lookahead auction, to distinguish it from its natural extension for BIC auctions, which we call the Bayesian lookahead auction. However, we are not able to show general revenue guarantee for the latter. (See Section 5 for a discussion.)

Lagrangian Relaxations. Proving revenue approximation results via partial Lagrangians was promoted by Cai et al. [4]. We review the basic idea here. Consider any constrained optimization problem:

$$(2.1) \quad \begin{aligned} \text{maximize:} \quad & f(x) \\ \text{subject to:} \quad & g_\alpha(x) \geq 0, \quad \forall \alpha \in A \\ & x \in \mathcal{C} \end{aligned}$$

Let μ be any nonnegative measure on A. The Lagrangian of (2.1) is $\mathcal{L}(x, \lambda) \coloneqq f(x) + \int_A g_\alpha(x) \lambda(\alpha) \, \mathrm{d}\mu(\alpha)$, where $\lambda \in \mathbb{R}^A$ are called the Lagrange multipliers, or the dual variables. The following fact is immediate.

FACT 2.1. *Let $\mathcal{C}'$ be $\mathcal{C} \cap \{x \mid g_\alpha(x) \geq 0, \forall \alpha \in A\}$ and assume $\mathcal{C}' \neq \emptyset$. Then for any $\lambda \geq 0$, the optimal value of (2.1) is bounded above by $\sup_{x \in \mathcal{C}'} \mathcal{L}(x, \lambda)$.*

Lagrangifying only some constraints (here $g_\alpha(x) \geq 0$) while leaving others (here $x \in \mathcal{C}$) proves often convenient for $\mathcal{L}$ to be used as a benchmark to be approximated. In revenue maximization, often the IC and IR constraints are Lagrangified, while the feasibility constraints are left aside.

Equal Revenue Distributions. Our large gap examples make use of the so-called (truncated) equal revenue distributions, which are standard distributions with an unbounded gap between the surplus $\mathbf{E}[v]$ and the optimal revenue extractable.

DEFINITION 2.4. (EQUAL REVENUE DISTRIBUTION) *An equal revenue distribution truncated at h is supported on $[1, h]$, with cumulative density function*

$$F(v) = \begin{cases} 1 - \frac{1}{v}, & \text{for } v \in [1, h); \\ 1, & \text{if } v = h. \end{cases}$$

It is easy to verify that $\mathbf{E}[v] \approx \log h$, whereas any posted price extracts revenue of only 1. Note that the equal revenue distribution is regular.

3 Single-Buyer Pricing with Signals

In this section we consider the problem of single-buyer single-item pricing with signals, and show our first main result:

THEOREM 3.1. *In the single-buyer pricing problem with signals, suppose the distribution is jointly regular and*

there is no negative payment, then the revenue achievable by a mechanism with private signals is bounded by 3 times the revenue extractable by naïve pricing, or mechanisms with public signals. Moreover, if the distribution is a mixture of k jointly regular distributions, the gap between the two is bounded by $3k$.

Before we give the proof in Section 3.2, it is instructive to see two examples, given in Section 3.1, which demonstrate that both conditions required by Theorem 3.1 (nonnegative payment and joint regularity) are necessary for the gap between private and public signal revenues to be finitely bounded.

3.1 Examples with Unbounded Value of Private Signals Our first example shows that the seller may make payments to the buyer to drastically amplify the power of private signals.

EXAMPLE 1. *Let the buyer's value v be drawn from an equal revenue distribution truncated at H (see Definition 2.4). For $\epsilon > 0$, let the signal s be equal to v with probability ϵ, and $s = *$ otherwise.*

Note that the distribution in this example is jointly regular. By the property of equal revenue distribution, with public signals, the revenue is at most $1 + \epsilon \ln H$. The next theorem shows that, with private signals, the seller can get revenue $\Omega((1 - \epsilon) \ln \ln H)$; when ϵ approaches 0 (i.e., the "informative" signals vanish), the latter is arbitrarily more in comparison. The mechanism extracting this revenue is nontrivial, with randomized allocations and, contrary to what may seem natural, pays the buyer when s is *informative*. Such negative payments entice the buyer to always report her true value.

THEOREM 3.2. *The optimal revenue with private signals is $\Omega((1 - \epsilon) \ln \ln H)$.*

Proof. We first describe the mechanism. Recall that we use x and p to denote allocation and payment rules. Let $x(v, *)$ be $\frac{\ln v}{\ln H}$, and $p(v, *) = vx(v, *) = \frac{v \ln v}{\ln H}$. Note that, when reporting her true value, the buyer receives utility 0 under the uninformative signal $s = *$. All incentives for truth-telling comes from when s is informative: when s is some value w, $x(v, w) = p(v, w) = 0$ for any $v \neq w$; when v and w agree, $x(v, w) = 1$, and the buyer receives a reward carefully chosen to satisfy the IC constraints: let $g(z)$ be $\max_y \frac{\ln y}{\ln H}(z - y)$; $p(v, v)$ is set to be $-\frac{f(v, *)}{f(v, v)} g(v)$, where f is the density function of the equal revenue distribution.

We relegate the proofs of the following two claims to Appendix A.

CLAIM 3.1. *The mechanism defined above is IC and ex post IR.*

CLAIM 3.2. *For all $z \geq 1$, $g(z) \leq \frac{z \ln z}{\ln H}$. If $z \geq e$, $g(z) \leq \frac{\ln z - \ln \ln z + 1}{\ln H} \cdot z$.*

We now calculate the expected revenue of the mechanism, which boils down to showing that the payment lost to the buyer under informative signals is far more than offset by the revenue gained under the uninformative signal:

$$
\begin{aligned}
\text{Rev} &= \int_1^H f(v, *)p(v, *) \, dv + \int_1^H f(v, v)p(v, v) \, dv \\
&= \int_1^H f(v, *)(p(v, *) - g(v)) \\
&\geq \int_e^H f(v, *)\left(\frac{\ln v}{\ln H} \cdot v - \frac{\ln v - \ln \ln v + 1}{\ln H} \cdot v\right) dv \\
&= (1 - \epsilon)\int_e^H \frac{1}{v} \cdot \frac{\ln \ln v - 1}{\ln H} \, dv \\
&= (1 - \epsilon) \cdot \frac{1}{\ln H} \cdot (\ln v \ln \ln v - 2 \ln v)\Big|_e^H \\
&\geq (1 - \epsilon) \cdot (\ln \ln H - 2),
\end{aligned}
$$

where the first inequality follows from Claim 3.2.

Our second example shows that, in a mechanism that make no negative payments, private signals can still blow up the revenue by an arbitrarily large factor if the supports of the conditional value distributions given different signals are allowed to interleave in arbitrary ways, a situation that would be precluded by joint regularity.

EXAMPLE 2. *Fix a large integer h. Signal s is uniformly distributed on $\{0, 1\}^h$. Conditioning on a realization of s, let the support of the conditional value distribution be $T_s := \{3k + s_k : k = 1, 2, \cdots, h\}$, and the value conditioning on s is drawn from the discrete equal revenue distribution supported on T_s, i.e.,*

$$
\mathbf{Pr}[v = t_i | s] = f(t_i | s) := \begin{cases} \frac{1}{t_i} - \frac{1}{t_{i+1}}, & i < h \\ \frac{1}{t_h}, & i = h, \end{cases}
$$

where $t_1 < t_2 < \ldots < t_h$ are the elements of T_s. $f(v|s)$ is 0 for $v \notin T_s$.

The optimal revenue under public signals is easily seen to be 1. Under private signals, we argue that the following mechanism is IC and extracts a revenue of $\Omega(\ln h)$: Let $x(v, s)$ be 1 if $f(v|s) > 0$, and 0 otherwise; let $p(v, s)$ be $\frac{1}{3}v$ if $f(v|s) > 0$, and 0 otherwise.

For any value v that the buyer has, telling the truth gets her utility $\mathbb{E}_s[x(v,s)\cdot v - p(v,s)] = \frac{2}{3}v$. By reporting any other value, the probability that her report would agree with the corresponding coordinate of s is only $\frac{1}{2}$, which means the probability of winning the item would be at most $\frac{1}{2}$, and her utility under deviation cannot be more than $\frac{1}{2}v$. This shows that the mechanism is IC. Its revenue is clearly a third of the buyer's expected value, which is $\Omega(\ln h)$.

REMARK 1. *In Example 2, the distribution is not jointly regular, despite its conditional value distribution resembling discretized equal revenue distributions. Once one tries to "pad" the gaps in the supports to make the density nonzero everywhere, the power of the private signals disappears. This remark is not meant to contrast discrete and continuous distributions; the point is that certain operations that are taken for granted in the absence of signals cannot be performed in correlated settings, due to the interaction among different signals.*

3.2 Proof of Theorem 3.1: Bounded Power of Private Signals under Regualrity and Nonnegative Payments We now prove Theorem 3.1. The optimal revenue by a mechanism with private signal and nonnegative payments is the value of the following optimzation problem:
(3.2)

$$
\begin{aligned}
\max: &\int_s \int_t f(t,s)p(t,s)\,\mathrm{d}t\,\mathrm{d}s \\
\text{s.t.:} &\int_s f(t,s)(t\cdot x(t,s) - p(t,s))\,\mathrm{d}s \\
&\geq \int_s f(t,s)(t\cdot x(t',s) - p(t',s))\,\mathrm{d}s \quad \forall t,t' \\
&t\cdot x(t,s) - p(t,s) \geq 0 \qquad\qquad\quad \forall t,s \\
&p(t,s) \geq 0 \qquad\qquad\qquad\qquad \forall t,s
\end{aligned}
$$

To prove Theorem 3.1, we will attempt to bound a partial Lagrangian relaxation of (3.2). To that end, let $\mathcal{L}(x,p,\lambda,\mu)$ be the Lagrangian relaxation defined as follows.

(3.3)
$$
\begin{aligned}
\mathcal{L}(x,p,\lambda,\mu) :=& \int_s \int_t f(t,s)p(t,s)\,\mathrm{d}t\,\mathrm{d}s \\
&+ \int_t \int_{t'} \lambda(t,t')\Big(\int_s f(t,s)\big(t\cdot(x(t,s)-x(t',s)) \\
&\qquad\qquad - (p(t,s)-p(t',s))\big)\,\mathrm{d}s\Big)\mathrm{d}t'\,\mathrm{d}t \\
&+ \int_s \int_t \mu(t,s)f(t,s)(t\cdot x(t,s) - p(t,s))\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

Without loss of generality, we will assume $x(t,s) = p(t,s) = 0$ whenever $f(t,s) = 0$ as the partial Lagrangian remains unchanged. Applying Fact 2.1, we

easily see that the value of (3.2) is upper bounded by $\inf_{\lambda,\mu\geq 0} \sup_{(x,p)\in\mathcal{F}} \mathcal{L}(x,p,\lambda,\mu)$.

Using Fubini and some rearrangement of (3.3) gives, whenever (x,p) is feasible,

$$
\begin{aligned}
&\mathcal{L}(x,p,\lambda,\mu) \\
&= \int_s \int_t p(t,s)\Big(f(t,s) - \int_{t'} f(t,s)\lambda(t,t')\mathrm{d}t' \\
&\qquad\qquad + \int_{t'} f(t',s)\lambda(t',t)\mathrm{d}t' - f(t,s)\mu(t,s)\Big)\mathrm{d}t\,\mathrm{d}s \\
&+ \int_s \int_t x(t,s)\Big(tf(t,s)\mu(t,s) + \int_{t'} tf(t,s)\lambda(t,t')\,\mathrm{d}t' \\
&\qquad\qquad - \int_{t'} t'f(t',s)\lambda(t',t)\,\mathrm{d}t'\Big)\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

A key step in this proof is to choose values for the dual variables λ and μ so that the resulting $\mathcal{L}$ acts as a meaningful upper bound that can be approximated by the optimal revenue with public signals.

We first argue that the values of μ's are determined by those of λ's. Since the only constraint on $p(t,s)$ is that of nonnegativity, the coefficient of any $p(t,s)$ should be nonpositive in order for $\mathcal{L}$ to be a finite bound. This gives a lower bound for each $\mu(t,s)$ whenever the values of λ's are fixed. The first double integral above involving the payments can therefore be removed, and only the second one involving the allocations gives $\mathcal{L}$ its value. Here, obviously $x(t,s)$ should be 1 if its coefficient is positive and 0 otherwise. It is easy to see that $\sup_{(x,p)\in\mathcal{F}} \mathcal{L}(x,p,\lambda,\mu)$ is non-decreasing in μ, and therefore μ should be equal to either 0 or the lower bound mentioned, whichever is larger. The problem therefore boils down to choosing values for the λ's.[2] Define

$$
\lambda^*(t,t') = \begin{cases} \frac{2}{t}, & t > 0, t' \leq t \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
g(t,s) :=& f(t,s) - \int_{t'} f(t,s)\lambda^*(t,t')\,\mathrm{d}t' \\
&+ \int_{t'} f(t',s)\lambda^*(t',t)\,\mathrm{d}t'.
\end{aligned}
$$

Then, as explained above, we may let $\mu^*(t,s)$ be such that $f(t,s)\mu^*(t,s) = [g(t,s)]_+$, where $[y]_+$ denotes $\max\{0,y\}$.

---

[2]Note that in what follows we crucially make use of the nonnegativity of the payments. If the payments were allowed to be negative, the coefficients of $p(t,s)$ would have to be 0, which means each μ would be equal to what previously was a lower bound. The nonnegativity constraint on μ therefore imposes constraints on the choice of λ's. In particular, our choice of λ's would have violated these constraints.

We introduce another shorthand notation to make it easy to write $\sup_{(x,p)\in\mathcal{F}}\mathcal{L}(x,p,\lambda^*,\mu^*)$:

$$h(t,s) := \int_{t'} tf(t,s)\lambda^*(t,t')\,\mathrm{d}t' - \int_{t'} t'f(t',s)\lambda^*(t',t)\,\mathrm{d}t'.$$

We can then bound $\mathcal{L}(x,p,\lambda^*,\mu^*)$ by

$$\mathcal{L}(x,p,\lambda^*,\mu^*)$$
$$\leq \int_s\int_t x(t,s)\Big(tf(t,s)\mu^*(t,s) + \int_{t'} tf(t,s)\lambda^*(t,t')\,\mathrm{d}t'$$
$$\qquad\qquad - \int_{t'} t'f(t',s)\lambda^*(t',t)\,\mathrm{d}t'\Big)\mathrm{d}t\,\mathrm{d}s$$
$$= \int_s\int_t x(t,s)\,([tg(t,s)]_+ + h(t,s))\,\mathrm{d}t\,\mathrm{d}s.$$

We will see in Lemma 3.2 that $x(t,s)h(t,s)$ is twice the virtual surplus of type t under signal s, and therefore is easy to be related to the revenue with public signals. On the other hand, $t\cdot g(t,s)$ differs from the *negative* of virtual value in a tantalizing way. It is intuitive that this difference is easier to bound than the expression itself; adding half of $h(t,s)$ has that effect. The following lemma paves the way towards this intuition. Its proof and all other missing proofs in this section can be found in Appendix B.

LEMMA 3.1. *The revenue of any mechanism with private signals and nonnegative prices is upper bounded by*

$$(3.4)\quad \int_s\int_t \left(\left[tg(t,s) + \frac{1}{2}h(t,s)\right]_+ + [h(t,s)]_+\right)\mathrm{d}t\,\mathrm{d}s.$$

Our choice of λ is independent of the signal and so we may bound the inner integral of (3.4) for each s. Let $\mathrm{DRev}(s)$ denote the contribution to the optimal revenue under public signals under signal s. The following lemma is the only place where we need the regularity assumption:

LEMMA 3.2. *If the distribution is jointly regular, then for all signals s,*

$$\int_t [h(t,s)]_+\,\mathrm{d}t \leq 2\cdot\mathrm{DRev}(s),$$
$$\int_t \left[t\cdot g(t,s) + \frac{1}{2}h(t,s)\right]_+\,\mathrm{d}t \leq \mathrm{DRev}(s).$$

Now we put together what we have shown so far: by Lemma 3.1, the revenue of any mechanism with private signal and nonnegative payments is bounded above by (3.4). Assuming joint regularity and applying Lemma 3.2, we have that the inner integral of (3.4) is bounded above by $3\,\mathrm{DRev}(s)$. Hence, the revenue of any

BSIC mechanism is bounded above by $3\int_s \mathrm{DRev}(s)\mathrm{d}s = 3\,\mathrm{DRev}$.

When the distribution is a mixture of k jointly regular distributions, all arguments remain true except that the bounds in Lemma 3.2 degrade by a factor of k. We relegate the details to Appendix B.

4 Multi-bidder Auctions

In this section we consider the revenues of DSIC and BIC multi-bidder auctions with correlated values. Our examples in Section 3.1 can be easily translated to multi-bidder distributions, showing:

COROLLARY 4.1. *In multi-bidder auctions with correlated values, if the auctioneer is allowed to make payments to the bidders, or if the value distribution is not jointly regular, the revenue of the optimal BIC mechanism can be more than that of any DSIC mechanism by an arbitrarily large factor.*

For completeness, we give a proof for the corollary in Appendix C. The main result of this section is that the bound we gave in Theorem 3.1 for the single-buyer problem can be extended to multi-bidder auctions, with the loss of another constant factor. It is tempting to use Ronen [18]'s lookahead auction (see Definition 2.3) which reduces the multi-bidder problem to extracting revenue from the highest bidder. However, Ronen's proof does not directly apply to BIC mechanisms due to subtle reasons we discuss in Section 5. Instead, we are able to carry on a "lookahead type of argument" in the partial Lagrangian under no negative payment and regularity: almost coincidentally, the dual variables we constructed in the proof of Theorem 3.1 have a property needed for such an argument (Claim 4.1).

THEOREM 4.1. *In a multi-bidder auction with jointly regular distribution, the revenue of the lookahead auction is at least $\frac{1}{5}$ that of the optimal BIC mechanism not using negative payments. Furthermore, if the the distribution is a mixture of k jointly regular distributions, the bound is $\frac{1}{3k+2}$.*

Proof. Following an approach similar to Section 3.2, we show the following lemma, whose proof and other missing proofs from this section can be found in Appendix C.

LEMMA 4.1. *The revenue of the optimal BIC, ex post IR mechanism is at most*

$$\max_{\boldsymbol{x}\geq\boldsymbol{0}} \int_{\boldsymbol{t}} \sum_i x_i(\boldsymbol{t})\left\{[g_i(\boldsymbol{t})t_i]_+ + h_i(\boldsymbol{t})\right\}\,\mathrm{d}\boldsymbol{t},$$
$$s.t.\quad \sum_i x_i(\boldsymbol{t}) \leq 1, \qquad \forall\boldsymbol{t},$$

where

$$g_i(\boldsymbol{t}) = -f(\boldsymbol{t}) + 2 \int_{t'_i > t_i} \frac{f(t'_i, \boldsymbol{t}_{-i})}{t'_i} \, dt'_i,$$

$$h_i(\boldsymbol{t}) = 2f(\boldsymbol{t}_{-i}) \left[t_i f(t_i | \boldsymbol{t}_{-i}) - (1 - F(t_i | \boldsymbol{t}_{-i})) \right].$$

CLAIM 4.1. *For any $\boldsymbol{t}$ and i, $[g_i(\boldsymbol{t})t_i]_+ + h_i(\boldsymbol{t}) \le 2t_i f(\boldsymbol{t})$.*

Let $U_i \subseteq T$ denote the set of type profiles in which bidder i is considered the potential winner by the lookahead auction. Recall that $\boldsymbol{t} \in U_i$ implies that t_i is a highest type in $\boldsymbol{t}$. Let secmax$(\boldsymbol{t})$ denote the second highest type in vector $\boldsymbol{t}$. We further partition $L(\mu^*, \lambda^*)$:

$$L(\mu^*, \lambda^*)$$

$$\le \max_{\boldsymbol{x} \ge \boldsymbol{0}} \sum_i \left\{ \int_{\boldsymbol{t} \in U_i} \sum_{i'} x_{i'}(\boldsymbol{t})([t_{i'} g_{i'}(\boldsymbol{t})]_+ + h_{i'}(\boldsymbol{t})) \, d\boldsymbol{t} \right\}$$

(4.1)

$$\le \max_{\boldsymbol{x} \ge \boldsymbol{0}} \sum_i \int_{\boldsymbol{t} \in U_i} \sum_{i' \ne i} x_{i'}(\boldsymbol{t})([t_{i'} g_{i'}(\boldsymbol{t})]_+ + h_{i'}(\boldsymbol{t})) \, d\boldsymbol{t}$$

$$\max_{\boldsymbol{x} \ge \boldsymbol{0}} \sum_i \int_{\boldsymbol{t} \in U_i} x_i(\boldsymbol{t})([t_i g_i(\boldsymbol{t})]_+ + h_i(\boldsymbol{t})) \, d\boldsymbol{t}$$

$$\text{s.t.} \sum_i x_i(\boldsymbol{t}) \le 1.$$

Note that $\max_{i' \ne i}\{t_{i'}\} = $ secmax$(\boldsymbol{t})$ for all $\boldsymbol{t} \in U_i$. Hence, by Claim 4.1 and that $\sum_{i' \ne i} x_{i'}(\boldsymbol{t}) \le 1$, we can bound the first term of (4.1) by

$$\sum_i \int_{\boldsymbol{t} \in U_i} 2 \max_{i' \ne i}\{t_{i'}\} f(\boldsymbol{t}) = 2 \int_{\boldsymbol{t}} \text{secmax}(\boldsymbol{t}) f(\boldsymbol{t}) \, d\boldsymbol{t}$$

which is twice the revenue of the second price auction. With slight modification of the argument in the proof of Theorem 3.1 (see Lemma C.1 in Appendix C), one can show that the second term of (4.1) is upper bounded by $3k$ times the revenue of the lookahead auction when the distribution is a mixture of k jointly regular distributions. This completes the proof of Theorem 4.1.

5 Discussion and An Open Question

In this work we give a fairly complete characterization, up to constant multiplicative factors, of revenue differences between disclosed and private side information when pricing for a single buyer, and of revenues differences between DSIC and BIC auctions for multiple bidders with correlated values. We find that regularity in the value distribution and the absence of negative payments are necessary and sufficient for these differences to be relatively small.

In connecting these two settings, we used an argument that is reminiscent of Ronen [18]'s proof for his

lookahead auction (Definition 2.3). However, our proof relies on the two mentioned conditions and our particular choice of dual variables. Ronen's original proof cannot be straightforwardly carried over for BIC auctions, and we briefly explain the reason here.

One property trivially true for DSIC auctions is convenient for showing the approximate optimality of the lookahead auction: given an optimal DSIC auction, if one were to zero out all the allocations and payments for all bidders except for the highest, the remaining auction is still DSIC. This is not necessarily true for BIC auctions. A bidder holding a certain value is sometimes the highest bidder, and sometimes not; her expected utility is affected by her allocations and payments when her value is not the highest. Removing these allocations and payments may invalidate the IC constraints. Therefore, by restricting an auction to only allocating to the highest bidders, it is unclear whether it would still be able to extract as much revenue as an unrestricted auction from the highest bidders.[3]

We therefore leave open the following interesting question: for multiple bidders with correlated values, can a BIC auction that allocates only to the highest bidders extract a constant fraction of the optimal BIC auction's revenue?

References

[1] Albert, M., Conitzer, V., and Lopomo, G. (2016). Maximizing revenue with limited correlation: The cost of ex-post incentive compatibility. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence*, AAAI'16, pages 376–382.

[2] Babaioff, M., Immorlica, N., Lucier, B., and Weinberg, S. M. (2014). A simple and approximately optimal mechanism for an additive buyer. In *FOCS*, pages 21–30. IEEE Computer Society.

[3] Brustle, J., Cai, Y., Wu, F., and Zhao, M. (2017). Approximating gains from trade in two-sided markets via simple mechanisms. In *EC*, pages 589–590. ACM.

[4] Cai, Y., Devanur, N. R., and Weinberg, S. M. (2016). A duality based unified approach to bayesian mechanism design. In *STOC*, pages 926–939. ACM.

[5] Cai, Y. and Zhao, M. (2017). Simple mechanisms for subadditive buyers via duality. In *STOC*, pages 170–183. ACM.

---

[3]This also calls to mind the comparison between β-exclusive and β-adjusted revenues considered by Yao [21] for multi-item auctions with bidders with additive values. It is also unclear whether Yao's argument can be applied here.

[6] Caragiannis, I., Kaklamanis, C., and Kyropoulou, M. (2016). Limitations of deterministic auction design for correlated bidders. *Transactions on Computation Theory*, 8(4):13:1–13:18.

[7] Chawla, S., Hartline, J. D., Malec, D. L., and Sivan, B. (2010). Multi-parameter mechanism design and sequential posted pricing. In *STOC*, pages 311–320. ACM.

[8] Crémer, J. and McLean, R. P. (1985). Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. *Econometrica*, 53(2):345–61.

[9] Crémer, J. and McLean, R. P. (1988). Full extraction of the surplus in bayesian and dominant strategy auctions. *Econometrica*, 56(6):1247–57.

[10] Devanur, N. R., Haghpanah, N., and Psomas, C. (2017). Optimal multi-unit mechanisms with private demands. In *EC*, pages 41–42. ACM.

[11] Dobzinski, S., Fu, H., and Kleinberg, R. (2015). Approximately optimal auctions for correlated bidders. *Games and Economic Behavior*, 92:349–369.

[12] Eden, A., Feldman, M., Friedler, O., Talgam-Cohen, I., and Weinberg, S. M. (2017a). The competition complexity of auctions: A bulow-klemperer result for multi-dimensional bidders. In *EC*, page 343. ACM.

[13] Eden, A., Feldman, M., Friedler, O., Talgam-Cohen, I., and Weinberg, S. M. (2017b). A simple and approximately optimal mechanism for a buyer with complements. In *EC*, page 323. ACM.

[14] Fiat, A., Goldner, K., Karlin, A. R., and Koutsoupias, E. (2016). The fedex problem. In *EC*, pages 21–22. ACM.

[15] Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):pp. 58–73.

[16] Papadimitriou, C. H. and Pierrakos, G. (2015). Optimal deterministic auctions with correlated priors. *Games and Economic Behavior*, 92:430–454.

[17] Riley, J. and Zeckhauser, R. (1983). Optimal selling strategies: When to haggle, when to hold firm. *The Quarterly Journal of Economics*, 98(2):267–289.

[18] Ronen, A. (2001). On approximating optimal auctions. In *EC*, pages 11–17. ACM.

[19] Sivan, B. and Syrgkanis, V. (2013). Vickrey auctions for irregular distributions. In *WINE*, volume 8289 of *Lecture Notes in Computer Science*, pages 422–435. Springer.

[20] Tang, P. and Wang, Z. (2016). Optimal auctions for negatively correlated items. In *EC*, pages 103–120. ACM.

[21] Yao, A. C. (2015). An n-to-1 bidder reduction for multi-item auctions and its applications. In *SODA*, pages 92–109. SIAM.

[22] Yao, A. C. (2017). Dominant-strategy versus bayesian multi-item auctions: Maximum revenue determination and comparison. In *EC*, pages 3–20. ACM.

A Proofs from Section 3.1

Proof. [Proof of Claim 3.1] Showing ex post IR is straightforward: for the informative signals, it is easy to see $p(v,w) \leq 0 \leq x(v,w) \cdot v$ for all v,w; for $s = *$, $p(v,*)$ equals $x(v,*) \cdot v$ by definition. We now show that mechanism is also IC. By reporting her true value, a buyer with value v has expected utility

$$\frac{1}{f(v,*) + f(v,v)}\big(f(v,*) \cdot 0 - f(v,v) \cdot p(v,v)\big)$$
$$= \frac{f(v,*)}{f(v,*) + f(v,v)} \cdot g(v).$$

By telling her value as w, the buyer would have expected utility

$$\frac{1}{f(v,*) + f(v,v)}\big(f(v,*) \cdot (v \cdot x(v,*) - p(v,*)) + f(v,v) \cdot 0\big)$$
$$= \frac{f(v,*)}{f(v,*) + f(v,v)} \cdot \frac{\ln w}{\ln H}(v - w) \leq \frac{f(v,*)}{f(v,*) + f(v,v)} \cdot g(v),$$

where the inequality holds by the definition of $g(z)$.

Proof. [Proof of Claim 3.2]
Let $h(y) = \ln y \cdot (z - y)$. Observe that when $y > z$, $h(y) < 0 \leq z \ln z$ and when $y \leq z$, $h(y) \leq z \ln z$, the first part of the claim is trivial. Now suppose $z \geq e$. Then $h'(y) = \frac{1}{y}(z - y) - \ln y$. Hence the maximum of $g(z)$ is achieved when $y(\ln y + 1) = z$. This implies $\ln y \leq \ln z - \ln \ln z + 1$, otherwise $y(\ln y + 1) > \frac{ez}{\ln z}(\ln z - \ln \ln z + 2) \geq z$. Therefore, $g(z) = \frac{\ln y}{\ln H}(z - y) \leq \frac{\ln z - \ln \ln z + 1}{\ln H} \cdot z$ when $z \geq e$.

B Proofs from Section 3.2

Proof. [Proof of Lemma 3.1] The optimal revenue is bounded above by

$$
\sup_{(x,p)\in\mathcal{F}} \mathcal{L}(x,p,\lambda^*,\mu^*)
$$

$$
\leq \sup_{(x,p)\in\mathcal{F}} \int_s \int_t x(t,s)\left([tg(t,s)]_+ + h(t,s)\right) \mathrm{d}t\,\mathrm{d}s
$$

$$
= \int_s \int_t [[tg(t,s)]_+ + h(t,s)]_+ \,\mathrm{d}t\,\mathrm{d}s.
$$

Let T_+ denote $\{(t,s) : g(t,s) \geq 0\}$ and let 1_{T_+} be its indicator function, then

$$
\int_s \int_t [[tg(t,s)]_+ + h(t,s)]_+ \,\mathrm{d}t\,\mathrm{d}s
$$

$$
= \int_s \int_t [tg(t,s) + h(t,s)]_+ 1_{T_+}\mathrm{d}t\,\mathrm{d}s
$$

$$
+ \int_s \int_t [h(t,s)]_+ 1_{\overline{T_+}}\mathrm{d}t\,\mathrm{d}s
$$

$$
\leq \int_s \int_t \left[tg(t,s) + \frac{1}{2}h(t,s)\right]_+ 1_{T_+}\,\mathrm{d}t\,\mathrm{d}s
$$

$$
+ \frac{1}{2}\int_s \int_t [h(t,s)]_+ 1_{T_+}\,\mathrm{d}t\,\mathrm{d}s + \int_s \int_t [h(t,s)]_+ 1_{\overline{T_+}}\,\mathrm{d}t\,\mathrm{d}s
$$

$$
\leq \int_s \int_t \left(\left[tg(t,s) + \frac{1}{2}h(t,s)\right]_+ + [h(t,s)]_+\right)\,\mathrm{d}t\,\mathrm{d}s.
$$

Proof. [Proof of Lemma 3.2] By definition, we have that $t \cdot g(t,s) = 2\int_{t'\geq t} \frac{t f(t',s)}{t'}\,\mathrm{d}t' - tf(t,s)$ and

$$
h(t,s) = 2tf(t,s) - 2\int_{t'\geq t} f(t',s)\,\mathrm{d}t'
$$

$$
= 2f(s)\left[tf(t|s) - 2[1 - F(t|s)]\right] = 2f(s,t)\varphi(t|s).
$$

where $f(s) := \int_{t'} f(t',s)\,\mathrm{d}t'$ is the marginal density of s, and $f(t|s)$, $F(t|s)$ and $\varphi(t|s)$ are the conditional density, conditional cumulative density and conditional virtual value of t given s, respectively. By Myerson's lemma (Lemma 2.1), $\int_t [h(t,s)]_+\,\mathrm{d}t$ is bounded by $2\,\mathrm{DRev}(s)$.

For the second part of the statement, we have

$$
\int_t \left[t\cdot g(t,s) + \frac{1}{2}h(t,s),\right]_+ \mathrm{d}t
$$

$$
= \int_t t\cdot\left[2\int_{t'\geq t} \frac{f(t',s)}{t'}\,\mathrm{d}t' - \int_{t'\geq t} \frac{f(t',s)}{t}\,\mathrm{d}t'\right]_+ \mathrm{d}t.
$$

We first show that there exists $0 \leq l \leq u$ such that the integrand is positive only on an interval $[l,u]$. Let $\psi_s(t)$

be $2\int_{t'\geq t} \frac{f(t',s)}{t'}\,\mathrm{d}t' - \int_{t'\geq t} \frac{f(t',s)}{t}\,\mathrm{d}t'$. Then

$$
\frac{\mathrm{d}\psi_s}{\mathrm{d}t}(t) = -2\frac{f(t,s)}{t} + \int_{t'\geq t} \frac{f(t',s)}{t^2}\,\mathrm{d}t' + \frac{f(t,s)}{t}
$$

$$
= -\frac{f(t,s)}{t^2}\left(t - \frac{1 - F(t|s)}{f(t|s)}\right)
$$

$$
= -\frac{f(t,s)}{t^2}\varphi(t|s).
$$

By joint regularity, $\varphi(t|s)$ is nondecreasing so $\psi_s(t)$ has at most two roots. In particular, there exists $0 \leq l \leq u$ such that $\psi_s(t) \geq 0$ if and only if $l \leq t \leq u$. For any $r \in \mathbb{R}_{\geq 0}$ we have

$$
\int_{t\leq r} t\cdot\psi_s(t)\,\mathrm{d}t
$$

$$
= \int_{t\leq r} 2\int_{t'\geq t} \frac{tf(t',s)}{t'}\,\mathrm{d}t'\,\mathrm{d}t - \int_{t\leq r}\int_{t'\geq t} f(t',s)\,\mathrm{d}t'\,\mathrm{d}t
$$

$$
= \int_{t'}\int_{t\leq\min\{r,t'\}} \frac{2tf(t',s)}{t'}\,\mathrm{d}t\,\mathrm{d}t'
$$

$$
- \int_{t'}\int_{t\leq\min\{r,t\}} f(t',s)\,\mathrm{d}t\,\mathrm{d}t'
$$

$$
= \int_{t'\leq r}\left(\int_{t\leq t'} \frac{2tf(t',s)}{t'}\,\mathrm{d}t - t'f(t',s)\right)\mathrm{d}t'
$$

$$
+ \int_{t'>r}\left(\int_{t\leq r} \frac{2tf(t',s)}{t'}\,\mathrm{d}t - rf(t',s)\right)\mathrm{d}t'
$$

$$
= \int_{t'>r}\left(\frac{r^2}{t'} - r\right)f(t',s)\,\mathrm{d}t' \leq 0.
$$

On the other hand, for any $r \in \mathbb{R}_{\geq 0}$,

$$
-\int_{t\leq r} t\cdot\psi_s(t)\,\mathrm{d}t = -\int_{t'>r}\left(\frac{r^2}{t'} - r\right)f(t',s)\,\mathrm{d}t'
$$

$$
\leq r\int_{t'\geq r} f(t',s)\,\mathrm{d}t' \leq \mathrm{DRev}(s).
$$

Thus, $\int_t [t\cdot g(t,s) + \frac{1}{2}h(t,s)]_+\mathrm{d}t = \int_{l\leq t\leq u} t\cdot\psi_s(t)\mathrm{d}t = \int_{t\leq u} t\cdot\psi_s(t)\,\mathrm{d}t - \int_{t\leq l} t\cdot\psi_s(t)\,\mathrm{d}t \leq \mathrm{DRev}(s)$.

Proof. [Proof for Mixture of k joingly regular distributions] Adapting the same flow variables $\lambda(\cdot,\cdot)$ as the proof for Theorem 3.1, we only need to prove the following variation of Lemma 3.2: if $f(\cdot,s)$ is a mixture of k regular distributions, that is, $f(t,s) = \sum_{i=1}^k \alpha_i f_i(t,s)$, where each f_i is the density of a jointly regular distribution, and $\sum_i \alpha_i = 1$, with $\alpha_i \geq 0, \forall i$, then

$$
\int_t [h(t,s)]_+\,\mathrm{d}t \leq 2k\cdot\mathrm{DRev}(s),
$$

$$
\int_t \left[t\cdot g(t,s) + \frac{1}{2}h(t,s)\right]_+ \mathrm{d}t \leq k\cdot\mathrm{DRev}(s).
$$

As before, we have

$$h(t,s) = 2tf(t,s) - 2\int_{t' \geq t} f(t',s)\,\mathrm{d}t',$$

$$\int_t \left[t \cdot g(t,s) + \frac{1}{2}h(t,s), \right]_+ \mathrm{d}t = \int_t t \cdot \left[2\int_{t' \geq t} \frac{f(t',s)}{t'}\,\mathrm{d}t' - \int_{t' \geq t} \frac{f(t',s)}{t}\,\mathrm{d}t' \right]_+ \mathrm{d}t.$$

Using the fact that $[x+y]_+ \leq [x]_+ + [y]_+$, we have

$$\int_t [h(t,s)]_+ \,\mathrm{d}t \leq \sum_{i \in [k]} 2\alpha_i \int_t f(t,s)[\varphi_s^i(t)]_+ \,\mathrm{d}t,$$

$$\int_t \left[t \cdot g(t,s) + \frac{1}{2}h(t,s), \right]_+ \,\mathrm{d}t$$
$$\leq \sum_{i \in [k]} \alpha_i \int_t t \cdot \left[2\int_{t' \geq t} \frac{f_i(t',s)}{t'}\,\mathrm{d}t' - \int_{t' \geq t} \frac{f_i(t',s)}{t}\,\mathrm{d}t' \right]_+ \,\mathrm{d}t,$$

where φ_s^i denotes the conditional virtual value of the i-th regular distribution given signal s. Now apply the argument in Section 3.2 to get

$$\int_t f(t,s)[\varphi_s^i(t)]_+ \,\mathrm{d}t \leq 2\,\mathrm{DRev}_i(s),$$

$$\int_t t \cdot \left[2\int_{t' \geq t} \frac{f_i(t',s)}{t'}\,\mathrm{d}t' - \int_{t' \geq t} \frac{f_i(t',s)}{t}\,\mathrm{d}t' \right]_+ \,\mathrm{d}t$$
$$\leq \mathrm{DRev}_i(s),$$

where $\mathrm{DRev}_i(s)$ denotes the part of the optimal DSIC revenue for distribution $f_i(\cdot, s)$ from signal s. Observing that $\alpha_i \,\mathrm{DRev}_i(s) \leq \mathrm{DRev}(s)$ for all i, we conclude the proof.

C Proofs from Section 4

Proof. [Proof of Corollary 4.1] Take either example from Section 3.1, we construct a two-bidder auction where the revenue of the optimal BIC mechanism is at least the revenue in the example under private signals, and revenue of the optimal DSIC mechanism is arbitrarily close to the optimal revenue under disclosed signals. In both examples, the signal space S is finite, and we can number the signals by integers so that $S = \{1, 2, \cdots, |S|\}$. For every (v, s) in the single-buyer example, let bidder 1's value be v, and bidder 2's value be $s \cdot \epsilon/|S|$ for an arbitrarily small $\epsilon > 0$. Take any single-buyer selling mechanism with private signals, use its allocation and payment rules for bidder 1 while taking bidder 2's value as the signal (while never

allocating or charging anything to bidder 2), then the resulting auction is BIC and extracts the same revenue as in the single-buyer example. On the other hand, if pricing under disclosed signals extracts at most revenue R from the single buyer, then any DSIC auction for the two-bidder setting can extract at most R from bidder 1, and at most ϵ from bidder 2 (because of individual rationality). The corollary follows by taking ϵ sufficiently small.

Proof. [Proof of Lemma 4.1] We consider the linear program that returns the revenue of the optimal BIC multi-bidder auction without negative payments.

$$\max_{\boldsymbol{x}, \boldsymbol{p}} \int_{\boldsymbol{t}} f(\boldsymbol{t}) \sum_i p_i(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

$$\text{s.t. } \int_{\boldsymbol{t}_{-i}} f(t_i, \boldsymbol{t}_{-i})[t_i \cdot x_i(t_i, \boldsymbol{t}_{-i}) - p_i(t_i, \boldsymbol{t}_{-i})]\,\mathrm{d}\boldsymbol{t}_{-i}$$
$$\geq \int_{\boldsymbol{t}_{-i}} f(t_i, \boldsymbol{t}_{-i})[t_i \cdot x_i(t_i', \boldsymbol{t}_{-i}) - p_i(t_i', \boldsymbol{t}_{-i})]\,\mathrm{d}\boldsymbol{t}_{-i} \quad \forall i, t_i, t_i'$$
$$t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t}) \geq 0, \quad \forall i, \boldsymbol{t}$$
$$p_i(\boldsymbol{t}) \geq 0, \quad \forall i, \boldsymbol{t}$$
$$\sum_i x_i(\boldsymbol{t}) \leq 1, \quad \forall \boldsymbol{t}$$
$$x_i(\boldsymbol{t}) \geq 0, \quad \forall i, \boldsymbol{t}$$

This revenue is upper bounded by the following partial Lagrangian, for any $\mu, \lambda \geq 0$:

$$L(\mu, \lambda)$$
$$:= \max_{\boldsymbol{x}, \boldsymbol{p}} \int_{\boldsymbol{t}} f(\boldsymbol{t}) \sum_i p_i(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$
$$+ \int_{\boldsymbol{t}} \sum_i \mu_i(\boldsymbol{t})[t_i \cdot x_i(\boldsymbol{t}) - p_i(\boldsymbol{t})]\,\mathrm{d}\boldsymbol{t}$$
$$+ \sum_i \int_{t_i} \int_{t_i'} \lambda_i(t_i, t_i') \Big\{ \int_{\boldsymbol{t}_{-i}} f(t_i, \boldsymbol{t}_{-i})[t_i \cdot x_i(t_i, \boldsymbol{t}_{-i})$$
$$- p_i(t_i, \boldsymbol{t}_{-i}) - t_i \cdot x_i(t_i', \boldsymbol{t}_{-i}) + p_i(t_i', \boldsymbol{t}_{-i})]\mathrm{d}\boldsymbol{t}_{-i} \Big\} \,\mathrm{d}t_i \mathrm{d}t_i'$$
$$= \max_{\boldsymbol{x}, \boldsymbol{p}} \sum_i \int_{\boldsymbol{t}} p_i(\boldsymbol{t}) \Big\{ f(\boldsymbol{t}) - \mu_i(\boldsymbol{t}) + \int_{t_i'} \Big[\lambda_i(t_i', t_i) f(t_i', \boldsymbol{t}_{-i})$$
$$- \lambda_i(t_i, t_i') f(t_i, \boldsymbol{t}_{-i}) \Big] \mathrm{d}t_i' \Big\} \mathrm{d}\boldsymbol{t}$$
$$+ \sum_i \int_{\boldsymbol{t}} x_i(\boldsymbol{t}) \Big\{ \mu_i(\boldsymbol{t})t_i + \int_{t_i'} \Big[t_i \cdot \lambda_i(t_i, t_i') f(t_i, \boldsymbol{t}_{-i})$$
$$- t_i' \cdot \lambda_i(t_i', t_i) f(t_i', \boldsymbol{t}_{-i}) \Big] \mathrm{d}t_i' \Big\} \mathrm{d}\boldsymbol{t}$$

s.t. $\quad p_i(\boldsymbol{t}) \geq 0, \qquad \forall i, \boldsymbol{t}$

$$\sum_i x_i(\boldsymbol{t}) \leq 1, \qquad \forall \boldsymbol{t}$$

$$x_i(\boldsymbol{t}) \geq 0, \qquad \forall i, \boldsymbol{t}$$

Similarly to the proof of Theorem 3.1, we try the following dual variables:

$$\lambda_i^*(t_i, t_i') = \begin{cases} \frac{2}{t_i}, & t_i > 0, t_i' \leq t_i; \\ 0, & \text{otherwise.} \end{cases}$$

Again, define

$$
\begin{aligned}
g_i(\boldsymbol{t}) :=& f(\boldsymbol{t}) - \int_{t_i'} f(\boldsymbol{t})\lambda_i^*(t_i, t_i')\, \mathrm{d}t' \\
& + \int_{t_i'} f(t_i', \boldsymbol{t}_{-i})\lambda_i^*(t_i', t_i)\, \mathrm{d}t_i' \\
=& -f(\boldsymbol{t}) + 2\int_{t_i' > t_i} \frac{f(t_i', \boldsymbol{t}_{-i})}{t_i'}\, \mathrm{d}t_i', \\
h_i(\boldsymbol{t}) =& \int_{t_i'} t_i \cdot \lambda_i^*(t_i, t_i')f(\boldsymbol{t})\, \mathrm{d}t_i' \\
& - \int_{t_i'} t_i' \cdot \lambda_i^*(t_i', t_i)f(t_i', \boldsymbol{t}_{-i})\, \mathrm{d}t_i' \\
=& 2f(\boldsymbol{t}_{-i})\left[t_i f(t_i | \boldsymbol{t}_{-i}) - (1 - F(t_i | \boldsymbol{t}_{-i})) \right].
\end{aligned}
$$

Let $\mu_i(\boldsymbol{t})$ be $[g_i(\boldsymbol{t})]_+$. Then

$$L(\mu^*, \lambda^*) \leq \max_{\boldsymbol{x} \geq \boldsymbol{0}} \int_{\boldsymbol{t}} \sum_i x_i(\boldsymbol{t}) \left\{ [g_i(\boldsymbol{t})t_i]_+ + h_i(\boldsymbol{t}) \right\}\, \mathrm{d}\boldsymbol{t},$$

$$\text{s.t.} \quad \sum_i x_i(\boldsymbol{t}) \leq 1, \qquad \forall \boldsymbol{t}.$$

Proof. [Proof of Claim 4.1] The claim is easily seen by a case analysis. For $g_i(\boldsymbol{t}) \leq 0$, $[g_i(\boldsymbol{t})t_i]_+ + h_i(\boldsymbol{t}) = h_i(\boldsymbol{t}) \leq 2f(\boldsymbol{t})t_i$, as $1 - F(t_i | \boldsymbol{t}_{-i}) \geq 0$. For $g_i(\boldsymbol{t}) \geq 0$, we have

$$
\begin{aligned}
& [g_i(\boldsymbol{t})t_i]_+ + h_i(\boldsymbol{t}) \\
=& t_i f(\boldsymbol{t}) + 2\left[\int_{t_i' > t_i} \frac{t_i f(t_i', \boldsymbol{t}_{-i})}{t_i'}\, \mathrm{d}t_i' - \int_{t_i' > t_i} \frac{t_i' f(t_i', \boldsymbol{t}_{-i})}{t_i'}\, \mathrm{d}t_i' \right] \\
\leq& t_i f(\boldsymbol{t}).
\end{aligned}
$$

LEMMA C.1. *Let* Rev_i *be the revenue extracted from bidder i by the lookahead auction, then*

$$\max_{0 \leq x_i(\boldsymbol{t}) \leq 1} x_i(\boldsymbol{t}) \int_{\boldsymbol{t} \in U_i} ([t_i g_i(\boldsymbol{t})]_+ + h_i(\boldsymbol{t}))\, \mathrm{d}\boldsymbol{t},$$

is bounded by $3\,\mathrm{Rev}_i$, *where* g_i *and* h_i *are defined as in (4.1),*

Proof. By the same argument as in the proof of Lemma 3.1, we have

$$\max_{0 \leq x_i(\boldsymbol{t}) \leq 1} x_i(\boldsymbol{t}) \int_{\boldsymbol{t} \in U_i} ([t_i g_i(\boldsymbol{t})]_+ + h_i(\boldsymbol{t}))\, \mathrm{d}\boldsymbol{t}$$

$$\leq \int_{\boldsymbol{t} \in U_i} [t_i g_i(\boldsymbol{t}) + \frac{1}{2}h_i(\boldsymbol{t})]_+ + [h_i(\boldsymbol{t})]_+\, \mathrm{d}\boldsymbol{t}.$$

By the same calculation as in the proof of Lemma 3.2, $h_i(\boldsymbol{t})$ is $2f(\boldsymbol{t})\varphi_i(t_i | \boldsymbol{t}_{-i})$, where $\varphi_i(t_i | \boldsymbol{t}_{-i})$ is the virtual value of t_i in the distribution conditioning on $\boldsymbol{t}_{-i}$. By joint regularity, $\mathcal{D}(t_i | \boldsymbol{t}_{-i})$ is regular, and so $\mathcal{D}(t_i | \boldsymbol{t}_{-i}, t_i > \max_{i' \neq i} t_{i'})$ is also regular. Therefore

$$\int_{\boldsymbol{t} \in U_i} [h_i(\boldsymbol{t})]_+\, \mathrm{d}\boldsymbol{t} = 2\,\mathrm{Rev}_i.$$

As for the term $[t_i g_i(\boldsymbol{t}) + \frac{1}{2}h_i(\boldsymbol{t})]_+$, from the calculation in Lemma 3.2, we know that, fixing any $\boldsymbol{t}_{-i}$, $t_i g_i(\boldsymbol{t})$ is nonnegative only on an interval $[l, u]$, and

$$\int_{t_i \leq u : (t_i, \boldsymbol{t}_{-i}) \in U_i} \left(t_i g_i(\boldsymbol{t}) + \frac{1}{2}h_i(\boldsymbol{t}) \right)\, \mathrm{d}t_i \leq 0,$$

$$-\int_{t_i \geq l : (t_i, \boldsymbol{t}_{-i}) \in U_i} \left(t_i g_i(\boldsymbol{t}) + \frac{1}{2}h_i(\boldsymbol{t}) \right)\, \mathrm{d}t_i$$

$$\leq l \int_{t_i \geq l : (t_i, \boldsymbol{t}_{-i}) \in U_i} f(t_i, \boldsymbol{t}_{-i})\, \mathrm{d}t_i$$

$$(C.1) \quad \leq \max(l, \max_{i' \neq i} t_{i'}) \int_{t_i \geq l : (t_i, \boldsymbol{t}_{-i}) \in U_i} f(t_i, \boldsymbol{t}_{-i})\, \mathrm{d}t_i,$$

where the RHS of (C.1) is upper bounded by the revenue of the lookahead auction when the bidders other than i bid $\boldsymbol{t}_{-i}$. Therefore,

$$\int_{\boldsymbol{t} \in U_i} [t_i g_i(\boldsymbol{t}) + \frac{1}{2}h_i(\boldsymbol{t})]_+\, \mathrm{d}\boldsymbol{t} \leq \mathrm{Rev}_i.$$