Tighter Bounds for Facility Games

Pinyan Lu[1], Yajun Wang[1], and Yuan Zhou[2]

[1] Microsoft Research Asia
{pinyanl,yajunw}@microsoft.com
[2] Carnegie Mellon University
yuanzhou@cs.cmu.edu

Abstract. In one dimensional facility games, public facilities are placed based on the reported locations of the agents, where all the locations of agents and facilities are on a real line. The cost of an agent is measured by the distance from its location to the nearest facility.

We study the approximation ratio of social welfare for *strategy-proof* mechanisms, where no agent can benefit by misreporting its location. In this paper, we use the total cost of agents as social welfare function. We study two extensions of the simplest version as in [9]: two facilities and multiple locations per agent. In both cases, we analyze randomized *strategy-proof* mechanisms, and give the first lower bound of 1.045 and 1.33, respectively. The latter lower bound is obtained by solving a related linear programming problem, and we believe that this new technique of proving lower bounds for randomized mechanisms may find applications in other problems and is of independent interest.

We also improve several approximation bounds in [9], and confirm a conjecture in [9].

1 Introduction

In a facility game, a planner is building public facilities while agents (players) are submitting their locations. In this paper, we study the facility game in one dimension, i.e., the locations of the agents and the facilities are in the real line. Let the position reported by agent i be $x_i \in \mathcal{R}_i \subseteq \mathcal{R}$. Assume the number of agents is n and the number of public facilities available is k. A (deterministic) mechanism for the k-facility game is simply a function

$$f : \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n \to \mathcal{R}^k.$$

In this paper, we assume $\mathcal{R}_i = \mathcal{R}$ for all agents. The *cost* of an agent is the distance from its *true* location to the nearest facility. Let $\{l_1, l_2, \ldots, l_k\}$ be the set of locations of the facilities. The cost of agent i is $\text{cost}(\{l_1, \ldots, l_k\}, x_i) = \min_{1\le j\le k} |x_i - l_j|$. A randomized mechanism returns a distribution over $\mathcal{R}^k$. Then the cost of agent i is the *expected* cost over the distribution returned by the randomized mechanism.

An agent may misreport its location if it can reduce its own cost. A usual solution concept is *strategy-proofness*, which is also the focus of this paper. In

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a *strategy-proof* mechanism, no agent can unilaterally misreport its location to reduce its own cost. For $\mathbf{x} = \{x_1, x_2, \ldots, x_i, \ldots, x_n\} \in \mathcal{R}^n$, we define $\mathbf{x}_{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. A mechanism is *strategy-proof* if for any x_i and $x'_i \neq x_i$, $\text{cost}(f(\mathbf{x}_{-i}, x_i), x_i) \leq \text{cost}(f(\mathbf{x}_{-i}, x'_i), x_i)$. In other words, no matter what other agents' strategies are, one of the best strategies for agent i is reporting its true location. Our strategy-proof randomized mechanisms are defined by the *expected costs* of the agents.

The facility game problem has a rich history in social science literature. Consider the case that we are building one facility in a discrete set of locations (alternatives). Agents are reporting its preference for the alternatives. The renowned Gibbard-Satterthwaite theorem [6,10] showed that if the preference on the alternatives for each agent can be arbitrary, the only strategy-proof mechanisms are the dictatorships when the number of alternatives are greater than two.

In the facility game, however, the preferences on the facility locations are not arbitrary. In particular, agent i has a single preferred location x_i. When two locations are on the same side of x_i, agent i will always prefer the one closer to x_i. This kind of admissible individual preferences are defined as *single-peaked* preferences, which was first discussed by Black [3]. Since the Gibbard-Satterthwaite theorem does not hold with single-peaked preferences, the facility game admits a much richer set of strategy-proof mechanisms. Moulin [8] characterized the class of all strategy-proof mechanisms for *one-facility* game in the real line. (One unnecessary assumption in the proof is dropped by Barberà and Jakson [2], and Sprumont [12].) In particular, a *generalized median voter scheme* is sufficient to characterize all strategy-proof mechanisms. Interested readers may refer to the detailed survey by Barberà [1].

More recently, Procaccia and Tennenholtz [9] studied the facility game in a different perspective. They consider the facility game as a special case of the game theoretic optimization problems where the optimal social welfare solution is not strategy-proof. They treat the facility game in a broader concept of the games that payments are not allowed or infeasible. Such mechanism design problems without payments are rarely studied by computer scientists, except some special problems [11].

Procaccia and Tennenholtz studied strategy-proof mechanisms with provable approximation ratios on social welfare, when the optimal solution is not strategy-proof. For the simplest case of one facility, the median mechanism is both strategy-proof and optimal for social welfare. Then Procaccia and Tennenholtz studied two extensions: (1) there are two facilities; (2) each agent controls multiple locations (with one facility). In both cases, the optimal solutions are no longer strategy-proof in general. Therefore, it is interesting to study strategy-proof mechanisms with good approximation ratios for these extensions. This is also the focus of this paper. A strategy-proof mechanism has an approximation ratio of α if for every input instance, the social cost for the output of the mechanism is always at most α times the social cost for any solution.

We remark that, if payment is allowed, then the well-know Vickrey-Clarke-Groves (VCG) mechanism [13,4,7] will give both optimal and strategy-proof solutions for both extensions. However, in many real world scenarios, payment is not available as noted by Schummer and Vohra [11]. We focus on the strategy-proof mechanisms without money in this paper.

1.1 Our Result

We study the approximation ratios of social welfare for the strategy-proof mechanisms in the facility game with one or more facilities. The social welfare function we use is the *social cost*, i.e., the total cost of all agents. We focus on the approximation ratios for *social cost* of the strategy-proof mechanisms, where we improve most results in [9]. Furthermore, we also provide several novel approximation bounds which are not previously available. Table 1 summarizes our contribution.

Table 1. Our results are in bold. The numbers in brackets are previous results in [9] unless stated otherwise. (N/A means no previous known bound.)

	Two Facilities	Multi-Location Per Agent (One Facility)
Deterministic	UB: $(n-2)$ LB: **2**(1.5)	UB: (3 [5]) LB: (3 [5])
Randomized	UB: $\mathbf{n/2}$ $(n-2)$ LB: **1.045**(N/A)	UB: $\mathbf{3 - \frac{2\min_{j\in N} w_j}{\sum_{j\in N} w_j}}$ $(2 + \frac{\lvert w_1 - w_2\rvert}{w_1+w_2}$ for $n = 2$ only) LB: **1.33** (N/A)

The organization of the paper is as follows. In Section 2, we provide improved upper and lower bounds of both deterministic and randomized strategy-proof mechanisms for the two-facility game. In Section 3, we study the cases when each agent controls more than one location. We conclude our paper in Section 4 with several open problems.

2 The Two-Facility Game

In this section, we study strategy-proof mechanisms for the two-facility game. We first provide a better randomized mechanism achieving approximation ratio $n/2$ for social cost. The only previously known upper bound is $n-2$, which is from a deterministic mechanism. Then we study the lower bounds both for the deterministic and randomized cases. For deterministic mechanisms, the lower bound is improved to 2 from 1.5 in [9]. For randomized mechanisms, we provide the first non-trivial approximation ratio lower bound of 1.045.

2.1 A Better Randomized Mechanism

The following mechanism is inspirited by Mechanism 2 from [9]. However, our proof is different and much simpler.

Mechanism 1. See Figure 1 for reference. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be the reported locations of the agents. Define $\mathrm{lt}(\mathbf{x}) = \min\{x_i\}$, $\mathrm{rt}(\mathbf{x}) = \max\{x_i\}$ and $\mathrm{mt}(\mathbf{x}) = (\mathrm{lt}(\mathbf{x}) + \mathrm{rt}(\mathbf{x}))/2$. We further define the *left boundary* $\mathrm{lb}(\mathbf{x}) = \max\{x_i : i \in N, x_i \leq \mathrm{mt}(\mathbf{x})\}$ and the *right boundary* $\mathrm{rb}(\mathbf{x}) = \min\{x_i : i \in N, x_i \geq \mathrm{mt}(\mathbf{x})\}$. Let $\mathrm{dist}(\mathbf{x}) = \max\{\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x}), \mathrm{lb}(\mathbf{x}) - \mathrm{lt}(\mathbf{x})\}$. We set $\overline{\mathrm{lb}}(\mathbf{x}) = \mathrm{lt}(\mathbf{x}) + \mathrm{dist}(\mathbf{x})$ and $\overline{\mathrm{rb}}(\mathbf{x}) = \mathrm{rt}(\mathbf{x}) - \mathrm{dist}(\mathbf{x})$. The mechanism returns $(\mathrm{lt}(\mathbf{x}), \mathrm{rt}(\mathbf{x}))$ or $(\overline{\mathrm{lb}}(\mathbf{x}), \overline{\mathrm{rb}}(\mathbf{x}))$, each with probability 1/2.

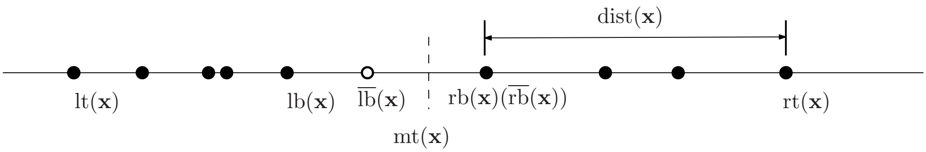

Fig. 1. Mechanism 1 picks $(\mathrm{lt}(\mathbf{x}), \mathrm{rt}(\mathbf{x}))$ or $(\overline{\mathrm{lb}}(\mathbf{x}), \overline{\mathrm{rb}}(\mathbf{x}))$, each with probability 1/2

Theorem 1. *Mechanism* 1 *is strategy-proof. The approximation ratio of Mechanism* 1 *is* $n/2$ *for social cost.*

Proof. We first prove the approximation ratio assuming that all agents report their true locations. By symmetry, we assume $\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x}) \geq \mathrm{lb}(\mathbf{x}) - \mathrm{lt}(\mathbf{x})$ as in Figure 1. Since we only have two facilities, either $\mathrm{lt}(\mathbf{x})$ and $\mathrm{rb}(\mathbf{x})$ or $\mathrm{rb}(\mathbf{x})$ and $\mathrm{rt}(\mathbf{x})$ are served by a same facility. Therefore the optimal solution is least $\min\{|\mathrm{lt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x})|, |\mathrm{rb}(\mathbf{x}) - \mathrm{rt}(\mathbf{x})|\} = \mathrm{dist}(\mathbf{x})$. On the other hand, for each agent, its expected cost is exactly $\mathrm{dist}(\mathbf{x})/2$ in this mechanism. So Mechanism 1 has an approximation ratio of $\frac{n}{2}$.

We then prove that Mechanism 1 is strategy-proof. We first show that any point other than the 3 points defining $\mathrm{lt}(\mathbf{x}), \mathrm{rt}(\mathbf{x})$ and $\mathrm{rb}(\mathbf{x})$ cannot benefit by misreporting its location. Let the new configuration be $\mathbf{x}'$. Consider the 3 points defining the previous $\mathrm{lt}(\mathbf{x}), \mathrm{rt}(\mathbf{x})$ and $\mathrm{rb}(\mathbf{x})$. No matter how the 3 points are partitioned by the new $\mathrm{mt}(\mathbf{x}')$, $\mathrm{dist}(\mathbf{x}') \geq \mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x})$, where $\mathbf{x}'$ is the new configuration. We know that the expected cost for any location in this configuration is at least $\mathrm{dist}(\mathbf{x}')/2$, which is at least as large as the honest cost $\mathrm{dist}(\mathbf{x}) = \mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x})$. The same argument also shows $\mathrm{lt}(\mathbf{x})$ (resp. $\mathrm{rt}(\mathbf{x})$) does not have incentive of reporting positions on the left (resp. right).

Consider the point $\mathrm{rb}(\mathbf{x})$. Its expected cost is $\frac{\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x})}{2}$ if it reports its true location. By lying, it cannot move the left or right boundary towards itself, and as a result, its expected cost in any new configuration is at least $\min\{|\mathrm{lt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x})|, |\mathrm{rb}(\mathbf{x}) - \mathrm{rt}(\mathbf{x})|\}/2 = (\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x}))/2$. Therefore, the point at $\mathrm{rb}(\mathbf{x})$ has no incentive to lie.

The only possible case left to analyze is that the agent at $\mathrm{lt}(\mathbf{x})$ (resp. $\mathrm{rt}(\mathbf{x})$) is reporting a location to the right (resp. left). Its expected cost is $(\mathrm{rt}(\mathbf{x}) - \overline{\mathrm{lb}}(\mathbf{x}))/2$ if it reports its true location. Reporting a location on its right can only move $\overline{\mathrm{lb}}(\mathbf{x}')$ toward right, which will hurt itself. Therefore the agent at $\mathrm{lt}(\mathbf{x})$ has no incentive to lie. Similar argument also holds if the agent at $\mathrm{rt}(\mathbf{x})$ reports its location on the left of $\mathrm{rt}(\mathbf{x})$.

To sum up, no agent has incentive to lie. Therefore, **Mechanism 1** is strategy-proof.

2.2 Lower Bounds

In this section, we show the approximation ratio lower bounds both for deterministic and randomized strategy-proof mechanisms. Both bounds are proved by the following construction, which is similar to the 1.5 lower bound example in [9].

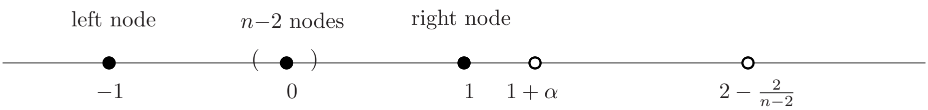

Fig. 2. Lower bound example for the two-facility game

Theorem 2 (Lower bound for deterministic mechanisms). *In the two-facility game, any deterministic strategy-proof mechanism $f : \mathcal{R}^n \to \mathcal{R}^2$ has an approximation ratio of at least $2 - \frac{4}{n-2}$ for social cost.*

Proof. See Figure 2 for the configuration. We have $n-2$ nodes at the origin and the left node at -1 and the right node at 1.

Assume to the contrary, there exists a strategy-proof mechanism with approximation ratio less than 2. Then this mechanism has to place one facility in the range $(-\frac{2}{n-2}, \frac{2}{n-2})$. Now consider the left node and the right node at -1 and 1. At least one of them is $1 - 2/(n-2)$ away from its closest facility. Without loss of generality, assume the right node at 1 is at least $1 - \frac{2}{n-2}$ away from the facilities.

If there is one facility on the right of 1, it must be placed at a position right to $2 - 2/(n-2)$ by our assumption. In this case, since the optimal cost is 1, the approximation ratio is at least $2 - \frac{4}{n-2}$ as one facility is always close to the origin.

Now consider the case that the closest facility to the right node at 1 is on the left. Let I be the image set of the closest facility to the right node when the right node moves and all other nodes remain fixed. Clearly, by strategy-proofness, $I \cap (\frac{2}{n-2}, 2 - \frac{2}{n-2}) = \emptyset$. On the other hand, $I \cap [2 - \frac{2}{n-2}, +\infty) \neq \emptyset$, otherwise the approximation ratio is unbounded when the right node moves to the infinity.

Take p as the left most point of $I \cap [2 - \frac{2}{n-2}, +\infty)$. ($p$ always exists, as I is a closed set.) If we place the right node at $p - 1 + \frac{2}{n-2}$, the closest facility to x is at p. Therefore, the cost of the mechanism for such a configuration is at least $2 - \frac{4}{n-2}$, as the other facility has to be close to the origin. Because the optimal cost is still 1, the approximation ratio is at least $2 - \frac{4}{n-2}$.

If the mechanism is randomized, the output is a distribution over $\mathcal{R}^2$. Notice that in a randomized mechanism, the cost of an agent is measured by the *expected distance* from its true location to the closest facility. We give the first non-trivial (greater than 1) approximation ratio lower bound of strategy-proof mechanisms for social cost in Theorem 3.

Theorem 3 (Lower bound for randomized mechanisms). *In a two-facility game, any randomized strategy-proof mechanism has an approximation ratio of at least* $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{n-2} \geq 1.045 - \frac{1}{n-2}$ *for the social cost for any* $n \geq 5$.

Proof. Again, we consider the point set as in Figure 2. Let the expected distance from -1, 0 and 1 to the closest facility be e_1, e_2 and e_3 respectively. Clearly, we have $e_1 + e_2 + e_3 \geq 1$. For any randomized strategy-proof mechanism with approximation ratio at most 2, $e_2 \leq \frac{2}{n-2}$. Without loss of generality, we assume $e_3 \geq \frac{1}{2} - \frac{1}{n-2}$.

Now we place the right node at 1 to a new position at $1+\alpha$ for some $\alpha \in (0, 1/2)$. Let e_3' be the expected distance from $1+\alpha$ to the nearest facility at the new configuration by the same strategy-proof mechanism. Because of strategy-proofness, $e_3' \geq \frac{1}{2} - \alpha - \frac{1}{n-2}$. (The condition $n \geq 5$ guarantees $e_3' \geq 0$ for the optimal α chosen later.)

Let $p(x)$ be the probability density function of the probability that the closest facility to the right node at $1+\alpha$ is at x in the new configuration. When $x \leq -\frac{1}{n-2}$, the closest facility is at *weighted distance* at least 1 to nodes at 0. When $x \geq \frac{1}{n-2}$, for any placement of the other facility, the sum of the *weighted* distances to the closest facility for the nodes at -1 and 0 is at least 1. In these two cases, the weighted distance to nodes at -1 and 0 is at least 1. Denote $P = \int_{-\frac{1}{n-2}}^{\frac{1}{n-2}} p(x)\,\mathrm{d}x$. Therefore, the total cost of the mechanism in the new configuration is at least:

$$\text{cost} \geq (1-P)\cdot 1 + e_3' \geq 1 + \frac{1}{2} - \alpha - \frac{1}{n-2} - P.$$

On the other hand, consider the distance to the node at $1+\alpha$. When the closest facility to $1+\alpha$ is $x \in (-\frac{1}{n-2}, \frac{1}{n-2})$, the total weighted distance from the nodes to the closest facilities is at least $1+\alpha$. Therefore, we have

$$\text{cost} \geq (1-P)\cdot 1 + P\cdot(1+\alpha) = 1 + \alpha \cdot P.$$

The optimal ratio is achieved when $P = \frac{1/2-\alpha-1/(n-2)}{1+\alpha}$ and the approximation ratio is at least

$$1 + \frac{1}{2} - \alpha - \frac{1}{n-2} - \frac{1/2-\alpha-1/(n-2)}{1+\alpha} \geq 1 + \frac{1}{2} - \frac{1}{n-2} - \frac{\alpha^2+1/2}{1+\alpha}.$$

Define $g(\alpha) = \frac{\alpha^2+1/2}{1+\alpha}$. The maximum ratio is achieved when $g'(\alpha) = 0$ with $\alpha = \frac{2-\sqrt{2}}{4}$, and the approximation ratio is at least $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{n-2}$.

Both lower bounds for deterministic and randomized strategy-proof mechanisms can be generalized to k facilities for $k \geq 3$. (Consider the configuration that two nodes on the two sides, and $k-1$ group of nodes in between. Each group of nodes (including the two singletons) are at unit distance away.) We have a direct corollary.

Corollary 1 (Lower bound for the k-facility game). *In the k-facility game for $k \geq 2$, any deterministic strategy-proof mechanism has an approximation ratio of at least $2 - \frac{4}{m}$ for the social cost, where $m = \lfloor \frac{n-2}{k-1} \rfloor$. Any* randomized strategy-proof mechanism *for the k-facility game has an approximation ratio of at least $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{m} \geq 1.045 - \frac{1}{m}$.*

3 Multiple Locations Per Agent

In this section, we study the case that each agent controls multiple locations. Assume agent i controls w_i locations, i.e., $\mathbf{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{iw_i}\}$. A (deterministic) mechanism with one facility in the multiple locations setting is a function $f : \mathcal{R}^{w_1} \times \cdots \times \mathcal{R}^{w_n} \to \mathcal{R}$ for n agents. Then, for agent i, its cost is defined as $\text{cost}(l, \mathbf{x}_i) = \sum_{j=1}^{w_i} |l - x_{ij}|$, where l is the location of the facility. As before, we are interested in minimizing the social cost of the agents, i.e., $\sum_{i \in N} \sum_{j=1}^{w_i} |l - x_{ij}|$, where $N = \{1, 2, \ldots, n\}$.

We first give a tight analysis of a randomized strategy-proof mechanism proposed in [9]. This in particular confirms a conjecture of [9]. Then we prove the first approximation ratio lower bound of 1.33 for any randomized truthful mechanism. This lower bound even holds for the simplest case that there are only two player and each controls the same number of locations. As pointed out by [9], our result here can be directly applied in the incentive compatible regression learning setting of Dekel et al.[5].

3.1 A Tight Analysis of a Randomized Mechanism

In [9], Procaccia and Tennenholtz proposed the following randomized mechanism in the setting of multiple locations:

Randomized Median Mechanism: Given $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, return $\text{med}(\mathbf{x}_i)$ with probability $w_i / (\sum_{j \in N} w_j)$.

If w_i is even, $\text{med}(\mathbf{x}_i)$ can either report the $\frac{w_i}{2}$th location or $\frac{w_i}{2}$+1-th location of $\mathbf{x}_i$. In [9], Procaccia and Tennenholtz gave a tight analysis for the case of two players($n = 2$), which has an approximation ratio of $2 + \frac{|w_1 - w_2|}{w_1 + w_2}$. They proposed as an open question for the bound in the general setting. In this section, we give a tight analysis of this randomized mechanism in the general setting, which in particular confirms the conjecture. Notice that $2 + \frac{|w_1 - w_2|}{w_1 + w_2} = 3 - \frac{2 \min_{j \in N} w_j}{\sum_{j \in N} w_j}$, when $n = 2$.

Theorem 4. *The* Randomized Median Mechanism *has an approximate ratio of $3 - \frac{2 \min_{j \in N} w_j}{\sum_{j \in N} w_j}$ for social cost.*

Proof. If $n = 1$, $\text{med}(x_1)$ is the optimal solution. So the mechanism has an approximate ratio of $3 - 2w_1/w_1 = 1$. Now we consider the case for $n \geq 2$.

Without loss of generality, we can reorder the players so that $\text{med}(\mathbf{x}_1) \leq \text{med}(\mathbf{x}_2) \leq \cdots \leq \text{med}(\mathbf{x}_n)$. Then it must be the case that $\text{med}(\mathbf{x}_1) \leq \text{med}(\mathbf{x}) \leq \text{med}(\mathbf{x}_n)$. The idea here is to construct a worst case instance for this mechanism and then analyze the approximate ratio for the worst case. Let i' be the largest i such that $\text{med}(\mathbf{x}_i) \leq \text{med}(\mathbf{x})$.

Claim. We can assume that the worst case satisfies the following properties: (1) w_i is even for all $i \in N$; (2) for all $i \leq i'$, $\text{med}(\mathbf{x}_i)$ returns the $\frac{w_i}{2}$-th point of $\mathbf{x}_i$; (3) and for all $i > i'$, $\text{med}(\mathbf{x}_i)$ returns the $(\frac{w_i}{2}+1)$-th point of $\mathbf{x}_i$.

We justify the claim as follows: if some w_i is odd, we can add one more point for agent i at the global median $\text{med}(\mathbf{x})$, then the original $\text{med}(\mathbf{x}_i)$ is still one of i-th two medians after adding the new point. We still return that value when we need to return $\text{med}(\mathbf{x}_i)$. After the modification, the expected cost can only increase while the optimal cost remain the same. So we can assume all w_i are even in a worst case. The properties (2) and (3) are obvious because returning the other point only improves the performance of the mechanism.

Now we assume that our instance satisfies all properties in Claim 1. By symmetry, we can further assume $\sum_{i=1}^{i'} w_i \geq \sum_{i=i'+1}^{n} w_i$. Let $W = \sum_{j\in N} w_j$ and $R(\text{med}(\mathbf{x}_i))$ be the rank of $\text{med}(\mathbf{x}_i)$ in the whole set $\mathbf{x}$. Let X be the ordered global set of $\mathbf{x}$ and X_i be the ith location in X. We perturb the points so that X_i and $R(\text{med}(\mathbf{x}_i))$ are well defined. Then for all $i \leq i'$, $R(\text{med}(\mathbf{x}_i)) \geq \sum_{j=1}^{i} \frac{w_i}{2}$; for all $i > i'$, $R(\text{med}(\mathbf{x}_i)) \leq W - \sum_{j=i}^{n} \frac{w_i}{2}$. The worst case happens when the above two sets of inequalities all reach equalities.

We further make the two sides more symmetric as follows. If $w_1 > w_n$, previously, the mechanism returns $X_{\frac{w_1}{2}}$ with probability $\frac{w_1}{W}$ and returns $X_{W+1-\frac{w_n}{2}}$ with probability $\frac{w_n}{W}$. We modify the mechanism by returning $X_{\frac{w_n}{2}}$ and $X_{W+1-\frac{w_n}{2}}$ both with probability $\frac{w_n}{W}$ and returning $X_{\frac{w_1}{2}}$ with probability $\frac{w_1-w_n}{W}$. We continue this process and finally we can get the following mechanism. There are $0 = k_0 < k_1 < k_2 < \cdots < k_m$ and $l \leq m$. The mechanism returns X_{k_i} and X_{W+1-k_i} both with probability $\frac{k_i-k_{i-1}}{k_m+k_l}$ if $1 \leq i \leq l$; returns X_{k_i} with probability $\frac{k_i-k_{i-1}}{k_m+k_l}$ if $l < i \leq m$. (The meaning of k_js are roughly $k_j = \sum_{i=1}^{j} \frac{w_i}{2}$. However due to the symmetrization process described above, we also have $k_j = \sum_{i=n}^{n-j} \frac{w_i}{2}$ for $j \leq l$.) We have $k_1 = \frac{\min\{w_1,w_n\}}{2} \geq \frac{\min_{j\in N} w_j}{2}$ and $k_m + k_l = W/2$.

To simply the notation, we define $\bar{i} = W+1-i$ and $K = k_m + k_l$. The optimal solution is $\text{OPT} = \sum_{i=1}^{W/2}(X_{\bar{i}} - X_i) \geq s = \sum_{j=1}^{m} a_j$, where $a_j = \sum_{i=k_{j-1}+1}^{k_j}(X_{\bar{i}} - X_i)$. Now we can compute the expected cost for this mechanism. For $1 \leq i \leq l$, we calculate the cost for X_{k_i} and $X_{\bar{k}_i}$ together. They both have probability $\frac{k_i-k_{i-1}}{k_m+k_l}$.

The cost for X_{k_i} is $\sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{\bar{j}} - X_{k_i}|)$. And we write that the cost for $X_{\bar{k}_i}$ as $\sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{\bar{k}_i}| + |X_{\bar{j}} - X_{\bar{k}_i}|)$. We combine the cost of X_{k_i} and $X_{\bar{k}_i}$ together.

$$2\sum_{j=1}^{i} a_j + 2\sum_{j=k_i+1}^{W/2} |X_{\bar{k}_i} - X_{k_i}|$$
$$= 2\sum_{j=1}^{i} a_j + 2(K-k_i)|X_{\bar{k}_i} - X_{k_i}| \le 2\sum_{j=1}^{i} a_j + 2\frac{a_i(K-k_i)}{k_j - k_{j-1}}$$

Now consider the case for $l+1 \le i \le m$. Similarly, the cost of X_{k_i} is

$$\sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2} (|X_j - X_{k_i}| + |X_{\bar{j}} - X_{k_i}|) \le \sum_{j=1}^{i} a_j + 2\frac{a_i(K-k_i)}{k_j - k_{j-1}}$$

Therefore the expected cost of the mechanism is no more than

$$\sum_{j=1}^{l} \frac{k_j - k_{j-1}}{K}(2\sum_{i=1}^{j} a_i + 2\frac{a_j(K-k_j)}{k_j - k_{j-1}}) + \sum_{j=l+1}^{m} \frac{k_j - k_{j-1}}{K}(\sum_{i=1}^{j} a_i + 2\frac{a_j(K-k_j)}{k_j - k_{j-1}})$$
$$\le \frac{1}{K}(2k_l\sum_{i=1}^{l} a_i + (k_m - k_l)\sum_{i=1}^{l} a_i + k_m\sum_{i=l+1}^{m} a_i - 2k_1\sum_{j=1}^{m} a_j) + 2s$$
$$= \frac{1}{K}(K\sum_{i=1}^{l} a_i + k_m\sum_{i=l+1}^{m} a_i - 2k_1\sum_{j=1}^{m} a_j) + 2s$$
$$\le (3 - \frac{2k_1}{K})s \le (3 - \frac{2\min_{j\in N} w_j}{\sum_{j\in N} w_j})\text{OPT}$$

The following corollary confirms a conjecture of [9] regarding the case where each agent controls the same number of locations.

Corollary 2. *If all the players control the same number of locations, the approximate ratio of* Randomized Median Mechanism *is* $3 - \frac{2}{n}$ *for social cost.*

3.2 Lower Bounds for Randomized Strategy-Proof Mechanisms

In this section, we consider the lower bound of the approximation ratios for randomized strategy-proof mechanisms in the multiple locations setting. We first give a 1.2 lower bound of the approximation ratio, based on a very simple instance. Then we extend to a more complicated instance, which we derive a lower bound of 1.33 by solving a linear programming instance.

Theorem 5. *Any randomized strategy-proof mechanism of the one-facility game has an approximation ratio at least* 1.2 *for social in the setting that each agent controls multiple locations.*

Proof. We assume to the contrary that there exists one strategy-proof mechanism M which has an approximate ratio $c < 1.2$. Consider the following three instances:

Instance 1: First player has 2 points at 0 and 1 point at 1; second player has 3 points at 1.
Instance 2: First player has 3 points at 0; second player has 3 points at 1.
Instance 3: First player has 3 points at 0; second player has 1 point at 0 and 2 points at 1.

Let P_1, P_2 and P_3 be the distribution of the facility the mechanism M gives for these three instances respectively. For all $x \in R$ and a distribution P on R, we use $\text{cost}(P, x)$ to denote $E_{y \sim P}|y - x|$. Then we have (for all $i = 1, 2, 3$)

$$\text{cost}(P_i, 0) + \text{cost}(P_i, 1) \geq 1.$$

We use $p_1(x), p_2(x)$ and $p_3(x)$ to denote the probability density function of P_1, P_2 and P_3 respectively. Let

$$\forall i \in \{1, 2, 3\},\ L_i = \int_{-\infty}^{0} -x p_i(x) \mathrm{d}x \quad \text{and} \quad R_i = \int_{1}^{+\infty} (x-1) p_i(x) \mathrm{d}x.$$

Now, we computer the cost of the players in each distribution. For the first player in Instance 1, its cost in distribution P_i is

$$\begin{aligned} & 2\text{cost}(P_i, 0) + \text{cost}(P_i, 1) = \text{cost}(P_i, 0) + (\text{cost}(P_i, 0) + \text{cost}(P_i, 1)) \\ &= \text{cost}(P_i, 0) + \int_{-\infty}^{+\infty} (|x| + |x-1|) p_i(x) dx = \text{cost}(P_i, 0) + 2L_i + 2R_i + 1 \end{aligned}$$

Since $L_1, R_1 \geq 0$, It's easy to see

$$\text{cost}(P_1, 0) \leq \text{cost}(P_1, 0) + 2(L_1 + R_1) \leq \text{cost}(P_2, 0) + 2(L_2 + R_2), \tag{1}$$

where the second inequality is because of the strategy-proofness (of the first player in Instance 1). By symmetry, we also have

$$\text{cost}(P_3, 1) \leq \text{cost}(P_2, 1) + 2(L_2 + R_2). \tag{2}$$

Using similar calculation as above, we can get the expected cost of Instance 1 as follows.

$$2\text{cost}(P_1, 0) + 4\text{cost}(P_1, 1) = 2\text{cost}(P_1, 1) + 2(2L_1 + 1 + 2R_1) \geq 2\text{cost}(P_1, 1) + 2.$$

Since the optimal cost is 2 and the approximate ratio is less than 1.2, we know that $\text{cost}(P_1, 1) + 2 < 2 \times 1.2 = 2.4$. Therefore, we have $\text{cost}(P_1, 1) < 0.2$ and hence $\text{cost}(P_1, 0) > 0.8$. Substituting the above inequality into (1), we get $\text{cost}(P_2, 0) + 2(L_2 + R_2) > 0.8$. Again by symmetry, we also have $\text{cost}(P_2, 1) + 2(L_2 + R_2) > 0.8$. Adding these two inequalities together, we have $\text{cost}(P_2, 0) + \text{cost}(P_2, 1) + 4(L_2 + R_2) > 1.6$. We also have $\text{cost}(P_2, 0) + \text{cost}(P_2, 1) = 1 + 2(L_2 + R_2)$. Substituting this, we get $L_2 + R_2 > 0.1$. On the other hand, note the approximate ratio condition of Instance 2 requires that $1 + 2(L_2 + R_2) < 1.2$. Thus we reach a contradiction.

To prove the lower bound of 1.33, we extend the above instances as follows. We employ $2K+1$ ($K \geq 1$ is an integer) instances (for $K=1$, this is exactly the same set of instances as above):

Instance i **(**$1 \leq i \leq K$**):** First player has $K+i$ points at 0 and $K+1-i$ points at 1; second player has all $2K+1$ points at 1.

Instance $K+1$**:** First player has all $2K+1$ points at 0; second player has all $2K+1$ points at 1.

Instance i **(**$K+2 \leq i \leq 2K+1$**):** First player has all $2K+1$ points at 0; second player has $i-K-1$ points at 0 and $3K+2-i$ points at 1.

Again, let P_i be the distribution of output of the mechanism on Instance i. Define the variables as $X_i = \text{cost}(P_i, 0)$ and $Y_i = \text{cost}(P_i, 1)$. Then, the strategy-proofness among the instances can be listed as linear constrains. Assume the approximation ratio is α. We want to compute the minimal ratio α so that all constrains are satisfied. It is then straightforward to formulate the following linear programming problem.

$$
\begin{aligned}
&\text{Minimize:}\quad \alpha \\
&\text{Subject to:} \\
&(K+i)X_i + (3K+2-i)Y_i \leq (K+i)\alpha, && 1 \leq i \leq K+1 \\
&(K+i)X_i + (3K+2-i)Y_i \leq (3K+2-i)\alpha, && K+2 \leq i \leq 2K+1 \\
&(K+i)X_i + (K+1-i)Y_i \\
&\qquad \leq (K+i)X_{i+1} + (K+1-i)Y_{i+1}, && 1 \leq i \leq K \\
&(i-K-1)X_i + (3K+2-i)Y_i \\
&\qquad \leq (i-K-1)X_{i-1} + (3K+2-i)Y_{i-1}, && K+2 \leq i \leq 2K+1 \\
&X_i \geq 0, Y_i \geq 0, X_i + Y_i \geq 1, && 1 \leq i \leq 2K+1
\end{aligned}
$$

First two sets of constrains come from the approximate ratio constrain. The next two sets of constrains are enforced by strategy-proofness. And the last two sets of constrains are boundary conditions.

Choosing $K = 500$, we solve this LP problem by computer and the optimal value is greater than 1.33. Therefore, if we set the approximation ratio to 1.33, there is no feasible solution for the linear programming which implies no feasible strategy-proof mechanism for the instances. So we have an approximation lower bound of 1.33.

Theorem 6. *Any randomized strategy-proof mechanism of the one-facility game has an approximation ratio at least* 1.33 *in the setting that each agent controls multiple locations.*

The numerical computation suggests that the optimal value for this LP problem is close to $\frac{4}{3}$ when K is large. It would be interesting to give an analytical proof for a lower bound of $\frac{4}{3}$. We leave it as an open question.

4 Conclusion

In this paper, we study the strategy-proof mechanisms in facility games. We derive approximation bounds for such mechanisms for social cost both in the two-facility game and the multiple location setting. Our results improves several bounds previously studied [9]. We also obtain some new approximation ratio lower bounds.

There are still a lot of interesting open questions. For example, in the two-facility game, the deterministic mechanism has an approximation ratio of $n - 2$ for social cost, while the lower bound is only 2. In randomized case, there is also a huge gap between $n/2$ and 1.045.

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